VI. *On Ellipsoidal Harmonics.*

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1. THE theory of Ellipsoidal Harmonics is usually based upon solutions of LAPLACE'S Equation suitably expressed in terms of ellipsoidal coordinates as independent variables. This manner of treating the subject, introduced by LAMÉ, has received its most complete exposition at the hands of HEINE in his 'Kugelfunctionen' and is probably the most direct and effective for all practical purposes. The Cartesian forms of the harmonics seem, however, to possess many advantages in point of clearness and suggestiveness. I have therefore been led by a perusal of GREEN'S Memoir on Ellipsoids of varying densities and by THOMSON and TAIT'S investigation of Spherical Harmonics to attempt the development of the subject by Cartesian processes.

In dealing with the general case of the ellipsoid of three unequal axes it is convenient to consider, in the first place (§§ 2–15), the propositions and theorems which relate to the forms applicable to the inside, and afterwards (§§ 16–21) those applicable to the outside of the ellipsoid. The main feature of my investigation of internal harmonics is the establishment of relations between the harmonics of the ellipsoid and certain harmonics of the sphere which have a close correspondence with them. By means of these relations it is possible, in the first place, to obtain readily the Cartesian forms of ellipsoidal harmonics (§ 15), and, in the next place, to determine an expansion, in ellipsoidal harmonics, having arbitrarily assigned values at the surface of the ellipsoid (§§ 8–10). As a particular case, I have taken the arbitrarily assigned value to be a homogeneous function of the coordinates x, y, z (§ 11), and it is an easy transition from this case to that of any function capable of expansion by TAYLOR'S theorem in ascending powers of x, y, z (§§ 12, 13). I have also expanded the reciprocal of the distance between two points, one of which is on the surface (§ 14).

The leading proposition in External Harmonics, upon which much of the work of this paper is based, relates to the expression of those harmonics in terms of differential operations upon the potential at an external point due to an ellipsoid of variable density. The proposition referred to, while possessing some peculiarities and difficulties of its own, is obtained in a manner analogous to that employed by CLERK MAXWELL in finding from the Theory of Attractions a physical interpretation of a Spherical Harmonic (§§ 17–20).

As illustrations of the use of the harmonics pertaining to an ellipsoid of three

unequal axes I have considered briefly the problem of the magnetism induced in a body of ellipsoidal form (§§ 22–25). In §§ 25–27 the potential has been found at an external point, due to a thin shell, bounded by similar and similarly situated ellipsoids, the density of which varies inversely as the cube of the distance from a fixed point, while § 28 contains a proposition in electrical capacity.

In §§ 30–52 I have investigated at some length various inside and outside forms of harmonics applicable to prolate and oblate spheroids, deducing them from the more general expressions of the earlier parts of the paper. The corresponding spherical harmonics, to which reference has already been made, reduce in the case of spheroids to the ordinary canonical conjugate system (§§ 30–32). From this system the next step is to investigate the forms of the internal harmonics of prolate spheroids (§§ 34–36), and the expressions for the external harmonics are then discussed in §§ 37–42. The corresponding forms for oblate spheroids are stated in §§ 43–49. In § 50 I have shown how the harmonics of the prolate spheroid reduce to those of the circular cylinder, and in § 51 to those of the paraboloid of revolution. The remainder of the paper is occupied with proofs of additional theorems for ellipsoidal harmonics, spherical harmonics of the second kind, and BESSEL'S functions.

Notation and Preliminary Investigations.

2. The notation employed by Dr. FERRERS in Chapter VI. of his 'Treatise on Spherical Harmonics,' besides being well known, seems so suitable that it will be convenient to adopt it as far as possible in the following investigations.

The semi-axes of the ellipsoid of reference being a, b, c, the ellipsoidal coordinates of any point xyz will be denoted by $\epsilon v v'$, so that

$$x^2 = \frac{(a^2+\epsilon)\,(a^2+v)\,(a^2+v')}{(a^2-b^2)\,(a^2-c^2)},$$

with similar expressions for y^2 and z^2.

The following quantities and operators will be of so frequent occurrence that special abbreviations are necessary.

$$\begin{array}{lll}
\dfrac{x^2}{a^2+\theta_r}+\dfrac{y^2}{b^2+\theta_r}+\dfrac{z^2}{c^2+\theta_r} & \text{will be denoted by} & \mathrm{K}_{\theta_r} \text{ or } \mathrm{K}_r. \\
\mathrm{K}_r - 1 \quad \ldots\ldots\ldots & \quad\text{,,} \qquad\quad \text{,,} & \Theta_r. \\
a^2\dfrac{\partial^2}{\partial x^2}+b^2\dfrac{\partial^2}{\partial y^2}+c^2\dfrac{\partial^2}{\partial z^2} \quad . & \quad\text{,,} \qquad\quad \text{,,} & \mathrm{D}^2. \\
\dfrac{1}{a^2+\theta_r}\dfrac{\partial^2}{\partial x^2}+\dfrac{1}{b^2+\theta_r}\dfrac{\partial^2}{\partial y^2}+\dfrac{1}{c^2+\theta_r}\dfrac{\partial^2}{\partial z^2} & \qquad\qquad\quad \text{,,} & \mathrm{E}_{\theta_r}{}^2 \text{ or } \mathrm{E}_r{}^2. \\
\dfrac{a^2}{a^2+\theta_r}\dfrac{\partial^2}{\partial x^2}+\dfrac{b^2}{b^2+\theta_r}\dfrac{\partial^2}{\partial y^2}+\dfrac{c^2}{c^2+\theta_r}\dfrac{\partial^2}{\partial z^2} & \qquad\qquad\quad \text{,,} & \mathfrak{E}_r{}^2.
\end{array}$$

3. All the types of interior harmonics subject to certain restrictions as to the values of θ, which will be presently investigated, are included under the following general scheme:—

$$\left\{\begin{array}{cccc} & x, & yz, & \\ 1, & y, & zx, & xyz \\ & z, & xy, & \end{array}\right\} \Theta_1\Theta_2 \ldots \Theta_n \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where any of the quantities inside the bracket may be taken as a multiplier of the product outside. The first column gives a harmonic of degree $2n$, the second $2n + 1$, the third $2n + 2$, the fourth $2n + 3$. Beginning with simple cases we see that Θ will satisfy LAPLACE'S equation provided

$$\frac{1}{a^2 + \theta} + \frac{1}{b^2 + \theta} + \frac{1}{c^2 + \theta} = 0.$$

Again, the function $\Theta_1\Theta_2$, when entered in LAPLACE'S equation, produces

$$\left(\frac{1}{a^2 + \theta_1} + \frac{1}{b^2 + \theta_1} + \frac{1}{c^2 + \theta_1}\right)\Theta_2 + \left(\frac{1}{a^2 + \theta_2} + \frac{1}{b^2 + \theta_2} + \frac{1}{c^2 + \theta_2}\right)\Theta_1$$
$$+ 4\left\{\frac{x^2}{(a^2 + \theta_1)(a^2 + \theta_2)} + \frac{y^2}{(b^2 + \theta_1)(b^2 + \theta_2)} + \frac{z^2}{(c^2 + \theta_1)(c^2 + \theta_2)}\right\} = 0.$$

Observing that the last line may be thrown into the form

$$4\,\frac{\Theta_2 - \Theta_1}{\theta_1 - \theta_2},$$

we see that, when LAPLACE'S equation is satisfied by the product $\Theta_1\Theta_2$, we must have

$$\left.\begin{array}{l} \psi_1 \equiv \dfrac{1}{a^2 + \theta_1} + \dfrac{1}{b^2 + \theta_1} + \dfrac{1}{c^2 + \theta_1} + \dfrac{4}{\theta_1 - \theta_2} = 0 \\ \psi_2 \equiv \dfrac{1}{a^2 + \theta_2} + \dfrac{1}{b^2 + \theta_2} + \dfrac{1}{c^2 + \theta_2} + \dfrac{4}{\theta_2 - \theta_1} = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The equations (2) we may call the characteristic equations of $\Theta_1\Theta_2$.

In like manner it can be proved that, if LAPLACE'S equation is satisfied by $\Theta_1\Theta_2 \ldots \Theta_n$, we must have

$$\left.\begin{array}{l} \psi_1 \equiv \dfrac{1}{a^2 + \theta_1} + \dfrac{1}{b^2 + \theta_1} + \dfrac{1}{c^2 + \theta_1} + \dfrac{4}{\theta_1 - \theta_2} + \ldots + \dfrac{4}{\theta_1 - \theta_n} = 0 \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \psi_n \equiv \dfrac{1}{a^2 + \theta_n} + \dfrac{1}{b^2 + \theta_n} + \dfrac{1}{c^2 + \theta_n} + \dfrac{4}{\theta_n - \theta_1} + \ldots + \dfrac{4}{\theta_n - \theta_{n-1}} = 0 \end{array}\right\} \quad . \quad (3).$$

The characteristic equations corresponding to $x\Theta_1 \ldots \Theta_n$ are the same as the equations $\psi_1, \ldots, \psi_n$ just found, with the exception of the terms in the first column which have to be multiplied by 3; *i.e.*, for all values of the suffix we have $3/(a^2 + \theta)$ in place of $1/(a^2 + \theta)$. Similar changes take place in the second and third columns if y and z are taken in place of x. Again, if the function be $yz\,\Theta_1 \ldots \Theta_n$, the characteristic equations are the same as in (3) with the exception of the second and third columns, which have each to be multiplied by 3; *i.e.*, we must have, for all values of the suffix, $3/(b^2 + \theta) + 3/(c^2 + \theta)$ instead of $1/(b^2 + \theta) + 1/(c^2 + \theta)$. Similar changes have to be made when the factors are zx and xy. Finally, in the case of $xyz\,\Theta_1 \ldots \Theta_n$, the first three columns are each to be multiplied by 3.

4. It should be remarked at this point that the system of expressions given by the scheme

$$\left\{\begin{matrix} & x, & yz, & \\ 1, & y, & zx, & xyz \\ & z, & xy, & \end{matrix}\right\} \mathrm{K}_1 \ldots \mathrm{K}_n \qquad (4)$$

are spherical harmonics for precisely the same values of θ which make the corresponding Θ expressions ellipsoidal harmonics.

The harmonics in the schemes (1) and (4) may be said to correspond when they involve the same values of θ.

The designation *type* will be used to denote any particular member of the group (1) or (4). The *degree* of a harmonic is the degree of its highest terms in x, y, z. Since there are $2n + 1$ independent harmonics of degree n (§ 5 below), if we suppose them arranged according to any definite system, the number indicating the position of any particular harmonic is its *order*. If, following CLERK MAXWELL, we denote a solid spherical harmonic of degree n and order σ by H_n^σ and suppose the type to be according to the scheme (4), the corresponding ellipsoidal harmonic or GREEN's harmonic of scheme (1) may be denoted by G_n^σ.

The outside harmonic corresponding to G_n^σ may be denoted by $\mathfrak{G}_n^\sigma$ so that at any external point xyz, $\mathfrak{G}_n^\sigma(xyz) = \mathrm{G}_n^\sigma\, \mathrm{I}_n^\sigma(xyz)$ where I_n^σ is an integral of the form

$$\int_\epsilon^\infty \frac{d\lambda}{(\theta_1 - \lambda)^2 (\theta_2 - \lambda)^2 \ldots \mathrm{L}}$$

in which $\theta_1, \theta_2, \ldots$ are the characteristics of G_n^σ and L is an abbreviation for $\sqrt{\{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)\}}$.

At the surface of the ellipsoid we may write $\mathrm{I}_n^\sigma(0)$ as the value of the integral.

When $n = 0$ the integral $\int_\epsilon^\infty d\lambda/\mathrm{L}$ (FERRERS's α) may be denoted by $\mathrm{I}_0(xyz)$, and at the surface of the ellipsoid by $\mathrm{I}_0(0)$.

5. In the equations (3) we observe that the first equation $\psi_1 = 0$ is of degree $n + 1$ in θ_1 and of degree 1 in all the other quantities $\theta_2 \ldots \theta_n$. The same is true of

the second in regard to θ_2 and so on. Hence all these quantities satisfy an equation whose degree is $n(n+1)$. But since the equations are such that if $n-1$ of the θ's be given the n^{th} is uniquely determined from them by each of $n-1$ equations, the roots must form $n+1$ groups of what we shall presently be enabled to describe as n conjugates. Each group gives rise to an ellipsoidal harmonic of the type $\Theta_1 \ldots \Theta_n$ so that there are, on the whole, $n+1$ independent varieties of this particular type. Since the introduction in one or more of the first three columns of equations (3) of the factor 3 does not affect this reasoning there will be $n+1$ distinct harmonics of any one of the types given by the general scheme (1).

6. Let $f(\theta) = 0$ be an equation of the n^{th} degree, satisfied by all the characteristics in (3) § 3, for any particular harmonic; then by the properties of roots the first equation may be thrown into the form

$$\frac{1}{a^2+\theta_1} + \frac{1}{b^2+\theta_1} + \frac{1}{c^2+\theta_1} + 2\frac{d^2 f(\theta_1)}{d\theta_1^2} \Big/ \frac{df(\theta_1)}{d\theta_1} = 0,$$

or, if the suffix be dropped, all the roots satisfy

$$\tfrac{1}{2}\{(b^2+\theta)(c^2+\theta)+\ldots\}\frac{df}{d\theta} + (a^2+\theta)(b^2+\theta)(c^2+\theta)\frac{d^2 f}{d\theta^2} = 0,$$

an equation whose degree is higher by 1 than $f(\theta) = 0$. The left-hand side must therefore be identically the same as

$$\{n(n+\tfrac{1}{2})\theta + r\}f(\theta)$$

where r is a constant to be determined.

The characteristics belonging to other types may be dealt with in a similar manner.

It is worthy of remark, as connecting the method of this paper with that expounded by FERRERS, that the equation just found may be thrown into the form

$$\sqrt{\{(a^2+\theta)(b^2+\theta)(c^2+\theta)\}}\frac{d}{d\theta}\sqrt{\{(a^2+\theta)(b^2+\theta)(c^2+\theta)\}}\frac{d}{d\theta}f = \{n(n+\tfrac{1}{2})\theta + r\}f,$$

and that all the equations for the other types may also be transformed into the same form. For instance, take the type $yz\,\Theta_1 \ldots \Theta_n$, and let $F(\theta) = 0$ be the equation of its characteristics. Put $\sqrt{\{(b^2+\theta)(c^2+\theta)\}}\,F(\theta) = f(\theta)$; then the differential equation for $f(\theta)$ will be of the same type as the above, the equation to determine r being however different and $n+1$ taking the place of n. (See FERRERS, VI., 6, 7, 8.)

Further consideration of the roots of the characteristic equation will most appropriately be taken when their positions and values have to be determined for ellipsoids of revolution.

Conjugacy of the K-*functions.*

7. All the ellipsoidal harmonics of the same degree are conjugates, in the sense that if G, G′ be any two of them, the integral $\iint GG'pdS$ is 0, where p is the perpendicular from the centre to the tangent plane of the element δS, and the integrations are taken over the ellipsoid. This has been proved by Dr. FERRERS in his Treatise VI., 14, and we shall deduce from it that $\iint HH'\,ds = 0$ where the integrations are taken over a concentric sphere.

For instance, let $G = \Theta_1 \ldots \Theta_n$ and $G' = \Phi_1 \ldots \Phi_n$; then, remembering that at the surface any factor Θ may be put equal to

$$-\theta\left\{\frac{x^2}{a^2(a^2+\theta)} + \frac{y^2}{b^2(b^2+\theta)} + \frac{z^2}{c^2(c^2+\theta)}\right\},$$

or,

$$-\theta\left(\frac{x_1{}^2}{a^2+\theta} + \frac{y_1{}^2}{b^2+\theta} + \frac{z_1{}^2}{c^2+\theta}\right),$$

where $x_1y_1z_1$ on a sphere of unit radius corresponds to xyz on the ellipsoid, we may put

$$G(x, y, z) = (-1)^n\,\theta_1 \ldots \theta_n H\,(x_1, y_1, z_1);$$
$$G'(x, y, z) = (-1)^n\,\phi_1 \ldots \phi_n H'(x_1, y_1, z_1);$$
$$pdS = abc\,ds.$$

Hence, $\iint HH'\,ds = 0$, the integrations being taken over the surface of the sphere.

This proves that the functions H, H′, corresponding to G, G′, are conjugate spherical harmonics, and the same proposition may readily be established for any two such functions whatever be their types.

The system of conjugates thus found, though possessing great generality on account of the endless variety of values which may be assigned to a, b, c, do not appear to take simple forms, except when two of the quantities, a, b, c, are equal, say $a = b$, in which case they reduce, as we shall afterwards see, to the canonical spherical forms. We can show that the poles, in CLERK MAXWELL's sense, of any one of the systems lie in one or other of the principal planes of the ellipsoid. Take the harmonic $K_1 \ldots K_n$ for example. It can readily be proved that, if $f(x, y, z)$ be a spherical harmonic of degree n, then

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = (-1)^n\frac{(2n)!}{2^n\,n!}\frac{f(x, y, z)}{r^{2n+1}}.$$

Hence

$$K_1 \ldots K_n = \frac{2^{2n}(2n)!}{(4n)!} r^{4n+1} \Pi \left(\frac{1}{a^2+\theta} \frac{\partial^2}{\partial x^2} + \ldots \right) \frac{1}{r}.$$

Now the n values of θ lie either between $-a^2$ and $-b^2$, or between $-b^2$ and $-c^2$ (see § 33 below). Suppose there are σ of the former and $n-\sigma$ of the latter. Then for the former values any single factor of the operator, in virtue of the relation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{dz^2} \right) \frac{1}{r}$$

being zero, becomes

$$-\frac{1}{c^2+\theta} \left\{ \frac{a^2-c^2}{a^2+\theta} \frac{\partial^2}{\partial x^2} + \frac{b^2-c^2}{b^2+\theta} \frac{\partial^2}{\partial y^2} \right\},$$

and for the latter

$$\frac{1}{a^2+\theta} \left\{ \frac{a^2-b^2}{b^2+\theta} \frac{\partial^2}{\partial y^2} + \frac{a^2-c^2}{c^2+\theta} \frac{\partial^2}{\partial z^2} \right\}.$$

Both of these operators can be put into real factors : there are, therefore, 2σ poles in the plane of xy, and $2n-2\sigma$ in the plane of yz.

Expansions having assigned Values at the surface of the Ellipsoid.

8. Let $v \equiv f(x, y, z)$ represent an assigned value at the surface of the ellipsoid, it is required to find a series of ellipsoidal harmonics which shall agree with it at the surface.

We may put $v = f(ax_1, by_1, cz_1)$, and an expansion in spherical harmonics can at once be found by LAPLACE's well-known formula, viz. :—

$$4\pi v = \iint v \, ds + \ldots + (2n+1) \iint v \, Q_n \, ds + \ldots$$

where Q_n is a zonal harmonic, whose pole is at $x_1y_1z_1$, and the integrations are taken over the sphere. This expansion has the value v at the surface of the sphere.

Since, according to the preceding article, the harmonics H_n^σ form a system of conjugates, Q_n may be expressed, as we shall presently see, in $2n+1$ terms of the form $A_n^\sigma \, H_n^\sigma (X, Y, Z) \, H_n^\sigma (x_1, y_1, z_1)$ where A_n^σ is a function of n and σ, and σ may have $(2n+1)$ different values. The point XYZ is the centre of the element ds.

Hence

$$4\pi v = \iint v \, ds + \ldots + (2n+1) \Sigma A_n^\sigma \, H_n^\sigma (x_1, y_1, z_1) \iint H_n^\sigma (X, Y, Z) \, v \, ds + \ldots$$

Or, if we express the series in terms of ellipsoidal harmonics and integrations over the ellipsoid

$$4\pi\, abc\, v = \iint vp\, d\mathrm{S} + \ldots + (2n+1)\, \Sigma\, \mathrm{B}_n^\sigma\, \mathrm{G}_n^\sigma(x, y, z) \iint \mathrm{G}_n^\sigma\, vp\, d\mathrm{S} + \ldots$$

In this expansion, xyz are the coordinates of the point at which it is required to express the value v in harmonics, and G_n^σ, v, p under the integral, are functions of the coordinates of the element $d\mathrm{S}$.

The quantities A_n^σ and B_n^σ, which are seen to be connected by the equation $\mathrm{A}_n^\sigma = \{\Pi(\theta)\}^2 \mathrm{B}_n^\sigma$, will now be determined.

Expansion of Q_n *in* K-*Functions.*

9. CLERK MAXWELL, in the chapter on "Spherical Harmonics" in the 'Electricity and Magnetism,' 2nd edition, vol. 1, p. 186, has introduced a theorem of great generality relating to any two spherical harmonics Y_n, Y_n' of the same degree n, viz., that

$$\iint \mathrm{Y}_n \mathrm{Y}_n'\, ds = \frac{4\pi}{2n+1} \frac{1}{n!} \frac{d^n}{dh_1 \ldots dh_n} \left(\mathrm{Y}_n' r^n\right)$$

where the integrations are taken over a sphere of radius unity, and the differentiations correspond to the n poles of the harmonic Y_n.

The result may be more conveniently stated thus: let $f(x, y, z)$ and $\mathrm{F}(x, y, z)$ be any two solid spherical harmonics of degree n; then

$$\iint f(x, y, z)\, \mathrm{F}(x, y, z)\, ds = 4\pi \frac{2^n n!}{(2n+1)!} f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \mathrm{F}(x, y, z).$$

This theorem offers a very simple criterion for the conjugacy of two functions, f and F; for, if they be conjugate, the right-hand side must vanish. Further, when a solid spherical harmonic of any degree is expressed in terms of a system of conjugates, each multiplied by a constant to be determined, the readiest way to determine the constant is to operate on both sides with the operator pertaining to the conjugate having that particular constant as multiplier. CLERK MAXWELL has pointed out this simple method of procedure in the chapter already cited (p. 200).

10. The pole of the harmonic Q_n being $x_1 y_1 z_1$ and XYZ, any other point on the sphere we may put

$$\mathrm{Q}_n(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$$
$$= \frac{(2n)!}{2^n n!\, n!} \left\{ (\mathrm{X}x_1 + \mathrm{Y}y_1 + \mathrm{Z}z_1)^n - \frac{n(n-1)}{2(2n-1)} (\mathrm{X}x_1 + \mathrm{Y}y_1 + \mathrm{Z}z_1)^{n-2} (\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2) + \ldots \right\}.$$

Put this equal to $\Sigma\, h_n^\sigma\, \mathrm{H}_n^\sigma(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ as we are entitled to do by § 7, and operate on both sides with

$$\mathrm{H}_n^\sigma\left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}}\right).$$

Now

$$\mathrm{H}_n^\sigma\left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}}\right) \mathrm{Q}_n(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = \mathrm{Q}_n\left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}}\right) \mathrm{H}_n^\sigma(\mathrm{X}, \mathrm{Y}, \mathrm{Z}),$$

$$= \frac{(2n)!}{2^n n!\, n!}\left(x_1 \frac{\partial}{\partial \mathrm{X}} + y_1 \frac{\partial}{\partial \mathrm{Y}} + z_1 \frac{\partial}{\partial \mathrm{Z}}\right)^n \mathrm{H}_n^\sigma(\mathrm{X}, \mathrm{Y}, \mathrm{Z}),$$

$$= \frac{(2n)!}{2^n n!\, n!} \mathrm{H}_n^\sigma\left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}}\right)(x_1 \mathrm{X} + y_1 \mathrm{Y} + z_1 \mathrm{Z})^n,$$

$$= \frac{(2n)!}{2^n n!} \mathrm{H}_n^\sigma(x_1, y_1, z_1).$$

The coefficient of h_n^σ is therefore determined, and we may write

$$\mathrm{Q}_n = \frac{(2n)!}{2^n n!} \overset{2n+1}{\Sigma} \frac{\mathrm{H}_n^\sigma(\mathrm{X}, \mathrm{Y}, \mathrm{Z})\, \mathrm{H}_n^\sigma(x_1, y_1, z_1)}{\mathrm{H}_n^\sigma\left(\frac{\partial}{\partial \mathrm{X}}, \frac{\partial}{\partial \mathrm{Y}}, \frac{\partial}{\partial \mathrm{Z}}\right) \mathrm{H}_n^\sigma(\mathrm{X}, \mathrm{Y}, \mathrm{Z})},$$

or, adopting the result of the preceding article,

$$\frac{4\pi}{2n+1} \overset{2n+1}{\Sigma} \frac{\mathrm{H}_n^\sigma(\mathrm{X}, \mathrm{Y}, \mathrm{Z})\, \mathrm{H}_n^\sigma(x_1, y_1, z_1)}{\iint (\mathrm{H}_n^\sigma)^2\, ds}.$$

Comparing this last with the assumed series in § 8 it appears that

$$\mathrm{A}_n^\sigma = \frac{4\pi}{2n+1} \frac{1}{\iint (\mathrm{H}_n^\sigma)^2\, ds} \quad \text{and} \quad \mathrm{B}_n^\sigma = \frac{4\pi}{2n+1} \frac{abc}{\iint (\mathrm{G}_n^\sigma)^2\, p\, d\mathrm{S}}.$$

The required expansion, having the value v at the surface of the ellipsoid, will therefore be

$$\frac{1}{4\pi abc} \iint vp\, d\mathrm{S} + \ldots . + \Sigma \frac{\mathrm{G}_n^\sigma(x, y, z) \iint \mathrm{G}_n^\sigma vp\, d\mathrm{S}}{\iint (\mathrm{G}_n^\sigma)^2\, p\, d\mathrm{S}} + \&c.$$

Case of a Homogeneous Function of the Coordinates.

11. Let $v \equiv f(x, y, z)$ be a homogeneous function of degree p.
If we write $u = f(\xi, \eta, \zeta)$, then will

$$v = \frac{1}{p!}\left(x \frac{\partial}{\partial \xi} + y \frac{\partial}{\partial \eta} + z \frac{\partial}{\partial \zeta}\right)^p u$$

$$= \frac{1}{p!}\left(ax_1 \frac{\partial}{\partial \xi} + by_1 \frac{\partial}{\partial \eta} + cz_1 \frac{\partial}{\partial \zeta}\right)^p u.$$

It will be convenient to expand this first by one of the formulæ of the preceding article in a series of K-Functions, viz. :—

$$v = \frac{1}{4\pi}\iint v\, ds + \ldots + \frac{(2n+1)!}{2^n n!}\Sigma \frac{\mathrm{H}_n^\sigma(x_1, y_1, z_1)\iint \mathrm{H}_n^\sigma v\, ds}{\mathrm{H}_n^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\mathrm{H}_n^\sigma(x, y, z)} + \ldots$$

In regard to the integral $\iint \mathrm{H}_n^\sigma v\, ds$ it is to be observed that p and n must both be even or both odd, otherwise the integral will manifestly vanish: let, therefore, $p = n + 2m$. In Part 1 of the 'Philosophical Transactions' for 1879, I have shown how to evaluate integrals of this class, and the result in the present instance may be presented in the form

$$h_n^m\, \mathrm{H}_n^\sigma\left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right)\mathrm{D}^{2m}u, \quad \text{where} \quad \mathrm{D}^2 = a^2\frac{\partial^2}{\partial\xi^2} + \ldots, (\S\, 2)$$

h_n^m being

$$\frac{4\pi}{(2n+2m+1)!}\frac{(m+n)!}{m!}2^m.$$

Reverting now to the expansion for v given above and observing that

$$\frac{(2n+1)!}{2_n n!}h_n^m \quad \text{reduces to} \quad \frac{4\pi}{2^m m!\,(2m+2n+1)(2m+2n-1)\ldots(2n+3)},$$

the required expansion may be presented under the form

$$\Sigma\Sigma\, \mathfrak{h}_n^m\,(m, n, \sigma)\, \mathrm{H}_n^\sigma(x_1, y_1, z_1)$$

where

$$(m, n, \sigma) = \frac{\mathrm{H}_n^\sigma\left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right)\mathrm{D}^{2m}u}{\mathrm{H}_n^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\mathrm{H}_n^\sigma(x, y, z)}$$

and

$$1/\mathfrak{h}_n^m = 2^m m!\,(2n+2m+1)\,(2n+2m-1)\ldots(2n+3).$$

The double Σ means that σ may have $2n+1$ different values for a given value of n, and n may have as many integral values as may be obtained from the equation $n = p - 2m$ by giving integral values to m. It only remains to express $\mathrm{H}_n^\sigma(x_1, y_1, z_1)$ in terms of $\mathrm{G}_n^\sigma(x, y, z)$, according to the relation of § 7.

Case of any Function capable of expansion by TAYLOR'S *Theorem.*

12. Consider now any function $v = f(\xi - x, \eta - y, \zeta - z)$ where x, y, z is on the surface, ξ, η, ζ outside, and let it be required to find a series of harmonics in x, y, z agreeing with v. If we assume that v may be expanded in ascending powers of x, y, z we may put

$$v = \Sigma (-1)^p \frac{1}{p!}\left(x\frac{\partial}{\partial\xi} + y\frac{\partial}{\partial\eta} + z\frac{\partial}{\partial\zeta}\right)^p u,$$

where $u = f(\xi, \eta, \zeta)$. The foregoing reasoning then shows that the coefficient of $H_n^\sigma (x_1, y_1, z_1)$, when we transform v into sphere coordinates, will be

$$(-1)^n \Sigma \mathfrak{h}_n^m (m, n, \sigma),$$

in which the Σ means that every possible integral value of m is to be taken from 0 upwards. In fact, if we put

$$u' = \left(1 + \frac{D^2}{2(2n+3)} + \frac{D^4}{2^2 2!(2n+5)(2n+3)} + \ldots\right) u,$$

the coefficient of $H_n^\sigma (x_1, y_1, z_1)$ will become

$$(-1)^n \frac{H_n^\sigma \left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right) u'}{H_n^\sigma \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) H_n^\sigma (x, y, z)}.$$

13. Should the function $f(x, y, z)$ satisfy LAPLACE'S equation, the expansion just found for $f(\xi - x, \eta - y, \zeta - z)$ will hold, not only at the surface of the ellipsoid, but at all points in its interior, and is, moreover, capable of simplification; for in that case

$$\frac{a^2 \frac{\partial^2}{\partial\xi^2}}{a^2 + \theta} + \ldots = -\theta\left(\frac{\frac{\partial^2}{\partial\xi^2}}{a^2+\theta} + \ldots\right),$$

and therefore

$$H_n^\sigma \left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right) = \Pi(-\theta)\, H_n^\sigma \left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right),$$

where $\Pi(-\theta)$ may include a factor of any of the forms a, b, c, bc, ca, ab, abc. Joining on the factor $\Pi(-\theta)$ to the harmonic H_n^σ, we finally obtain

$$f(\xi - x, \eta - y, \zeta - z) = \Sigma\Sigma(-1)^n \frac{\mathrm{H}_n^\sigma\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right) u'}{\mathrm{H}_n^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \mathrm{H}_n^\sigma(x, y, z)} \mathrm{G}_n^\sigma(x, y, z).$$

We may deduce from this result a practical rule for expressing any function of the form $f(\xi - x, \eta - y, \zeta - z)$, satisfying LAPLACE's equation in ellipsoidal harmonics. First express f in spherical K-conjugate functions, then substitute u' for u, and change H_n^σ into G_n^σ.

In the particular case, when the function is $\{(\xi - x)^2 + (\eta - y)^2 + (\zeta + z)^2\}^{-1/2}$ the above result may be put into a more concise form. The foregoing proof, however, depending as it does upon an expansion by TAYLOR's theorem, which will not be convergent unless $\xi\eta\zeta$ is further from the centre than xyz, can only be accepted without ambiguity when the point $\xi\eta\zeta$ is at a greater distance from the centre than the longest semi-axis. I shall, therefore, give another proof of the important case in question, against which this objection cannot be urged.

GREEN's *Function.*

14. Let E $(\xi\eta\zeta)$ and P (xyz) be two points, of which the former is outside of, and the latter is on, the ellipsoid, and let 1/EP be expanded in a series of ellipsoidal harmonics by the formula at the end of § 10, viz.:—

$$\frac{1}{4\pi abc}\iint \frac{1}{\mathrm{EP}}\, p\, d\mathrm{S} + \ldots + \Sigma \frac{\mathrm{G}_n^\sigma(x, y, z) \iint \frac{1}{\mathrm{EP}} \mathrm{G}_n^\sigma(f, g, h)\, p\, d\mathrm{S}}{\iint (\mathrm{G}_n^\sigma)^2\, p\, d\mathrm{S}} + \ldots$$

where f, g, h are the coordinates of the element $d\mathrm{S}$.

Now it can be easily shown that density $p\mathrm{G}_n^\sigma(f, g, h)$ on the surface produces potential at an outside point $\xi\eta\zeta$ equal to

$$2\pi abc\, \{\Pi(\theta)\}^2\, \mathfrak{G}_n^\sigma(\xi, \eta, \zeta.) \quad \text{(See § 4.)}$$

The expansion of 1/EP is therefore

$$\tfrac{1}{2}\mathrm{I}_0(\xi, \eta, \zeta) + \ldots + 2\pi \frac{\mathrm{G}_n^\sigma(x, y, z)\, \mathfrak{G}_n^\sigma(\xi, \eta, \zeta)}{\iint (\mathrm{H}_n^\sigma)^2\, ds} + \ldots$$

If $P(xyz)$ be now supposed to be outside of the ellipsoid, GREEN's function at P due to the electricity induced by unit charge at E, or at E due to unit charge at P, is

$$-\tfrac{1}{2}\left\{\frac{I_0(xyz)\,I_0(\xi,\eta,\zeta)}{I_0(0)}+\ldots+4\pi\frac{\mathfrak{S}_n^\sigma(\xi,\eta,\zeta)\,\mathfrak{S}_n^\sigma(x,y,z)}{I_n^\sigma(0)\iint(H_n^\sigma)^2\,ds}+\ldots\right\}.$$

Connexions between corresponding Ellipsoidal and Spherical Harmonics.

15. In § 13 it was shown that the expansion of a function satisfying LAPLACE's equation, in ellipsoidal harmonics, might be obtained from the expansion of the same function in corresponding spherical harmonics by the adoption of certain changes. To render this method complete, it is necessary to show how an internal ellipsoidal harmonic can be expressed most readily in a series of rational algebraic functions. The present article is intended to effect this, and the results obtained will be found useful in the case of ellipsoids of revolution.

If we denote $K_1 \ldots K_n$ by S_n and the sum of the products of the K's taken together r at a time by S_r then

$$\Theta_1 \ldots \Theta_n = (K_1-1)\ldots(K_n-1) = S_n - S_{n-1} + \&c.$$

It will now be shown how the S functions may all be derived from S_n by successive differentiation.

Theorem i. With the values of $\theta_1 \ldots \theta_n$ given in (3) § 3 corresponding to the functions $\Theta_1 \ldots \Theta_n$, if the operator D^2 be applied to S_r the result will be

$$(2n-2r+2)(2n+2r-1)\,S_{r-1}.$$

To show this let the result of the operation D^2 upon the product of two factors K_1K_2 be first found. We shall have

$$\left(\frac{2a^2}{a^2+\theta_1}+\frac{2b^2}{b^2+\theta_1}+\frac{2c^2}{c^2+\theta_1}\right)K_2+\left(\frac{2a^2}{a^2+\theta_2}+\frac{2b^2}{b^2+\theta_2}+\frac{2c^2}{c^2+\theta_2}\right)K_1$$
$$+\frac{8a^2x^2}{(a^2+\theta_1)(a^2+\theta_2)}+\frac{8b^2y^2}{(b^2+\theta_1)(b^2+\theta_2)}+\frac{8c^2z^2}{(c^2+\theta_1)(c^2+\theta_2)}.$$

Now the last line is readily seen to be equal to

$$\frac{8}{\theta_1-\theta_2}(\theta_1K_1-\theta_2K_2).$$

Hence the coefficient of K_2 is

$$\frac{2a^2}{a^2+\theta_1}+\frac{2b^2}{b^2+\theta_1}+\frac{2c^2}{c^2+\theta_1}-\frac{8\theta_2}{\theta_1-\theta_2}.$$

Let attention be now fixed upon one particular term in S_{r-1} after the operation D^2 is supposed to have been performed upon S_r, say the term $K_{n-r+2} \ldots K_n$. This term will obviously arise from differentiations upon $(K_1 + \ldots + K_{n-r+1}) K_{n-r+2} \ldots K_n$ and from none other. The resulting coefficient written at length will be

$$\begin{aligned}
&\sum_1^{n-r+1}\left(\frac{2a^2}{a^2+\theta_s} + \frac{2b^2}{b^2+\theta_s} + \frac{2c^2}{c^2+\theta_s}\right) \\
&- 8\,\frac{\theta_{n-r+2}}{\theta_1 - \theta_{n-r+2}} - \ldots - 8\,\frac{\theta_n}{\theta_1 - \theta_n} \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&- 8\,\frac{\theta_{n-r+2}}{\theta_{n-r+1} - \theta_{n-r+2}} - \ldots - 8\,\frac{\theta_n}{\theta_{n-r+1} - \theta_n}
\end{aligned}$$

Putting $a^2/(a^2+\theta) = 1 - \theta/(a^2+\theta)$ in each case, and bearing in mind the equations (3), § 3, we find

$$\begin{aligned}
&6(n-r+1) + 8(n-r+1)(r-1) \\
&+ 8\theta_1\left(\frac{1}{\theta_1 - \theta_2} + \ldots + \frac{1}{\theta_1 - \theta_{n-r+1}}\right) \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&+ 8\theta_{n-r+1}\left(\frac{1}{\theta_{n-r+1} - \theta_1} + \ldots + \frac{1}{\theta_{n-r+1} - \theta_{n-r}}\right).
\end{aligned}$$

The terms after the first line now combine in two's, the sum of each pair, involving the same two θ's, being 8. Hence the final result is

$$6(n-r+1) + 8(n-r+1)(r-1) + 4(n-r+1)(n-r).$$

$$= (2n - 2r + 2)(2n + 2r - 1).$$

The next three results are given without proof.

ii. For a function of the type $x\Theta_1 \ldots \Theta_n$ the operation $D^2 x\, S_r$ gives

$$(2n - 2r + 2)(2n + 2r + 1)\, x\, S_{r-1}.$$

iii. Of the type $yz\,\Theta_1 \ldots \Theta_{n-1}$, $D^2 yz\, S_{r-1} = (2n - 2r + 2)(2n + 2r - 1)\, yz\, S_{r-2}$.

iv. Of the type $xyz\,\Theta_1 \ldots \Theta_{n-1}$, $D^2 xyz\, S_{r-1} = (2n - 2r + 2)(2n + 2r + 1)\, xyz\, S_{r-2}$.

The equalities of the coefficients in i. and iii. and in ii. and iv. should not escape notice. They show that ellipsoidal harmonics of even and odd degrees may be expressed respectively as follows :—

$$G_{2n} = \left(1 - \frac{D^2}{2(4n-1)} + \frac{D^4}{2^2 2!\,(4n-1)(4n-3)} - \&c.\right) H_{2n}$$

$$G_{2n+1} = \left(1 - \frac{D^2}{2(4n+1)} + \frac{D^4}{2^2 2!\,(4n+1)(4n-1)} - \&c.\right) H_{2n+1}.$$

Both cases may be combined and the result exhibited in the form

$$G_n = \left(1 - \frac{D^2}{2(2n-1)} + \ldots + (-1)^m \frac{D^{2m}}{2^m m!\,(2n-1)(2n-3)\ldots(2n-2m+1)} + \ldots\right) H_n,$$

where the greatest value of m is $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, according as n is even or odd.

This result may be expressed concisely by means of a definite integral, viz.,

$$G_n = \frac{1}{(2n)!}\int_0^\infty (u^2 - D^2)^n e^{-u}\, du . H_n$$

or

$$\frac{1}{(2n)!}\int_0^\infty \left(\frac{d}{du}\right)^n (u^2 - D^2)^n e^{-u}\, du . H_n$$

$$= \frac{2^n n!}{(2n)!} D^{2n} \int_0^\infty P_n (u D^{-1})\, e^{-u}\, du . H_n,$$

P_n being a zonal harmonic of degree n.

Expansion of an External Ellipsoidal Harmonic in Spherical Harmonics.

16. Comparing the coefficient of $G_n^\sigma(x, y, z)$ in the expansion of

$$\{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\}^{-1/2}$$

in § 14 with that in § 13, we find

$$\frac{2\pi \mathfrak{G}_n^\sigma(\xi, \eta, \zeta)}{\iint (H_n^\sigma)^2\, ds} = \frac{(-1)^n H_n^\sigma\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right) u'}{H_n^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) H_n^\sigma(x, y, z)}$$

where

$$u' = \left\{1 + \frac{D^2}{2(2n+3)} + \ldots\right\} \frac{1}{\sqrt{(\xi^2 + \eta^2 + \zeta^2)}}, \quad D^2 = a^2 \frac{\partial^2}{\partial \xi^2} + \ldots$$

Hence

$$\mathfrak{G}_n^\sigma(\xi, \eta, \zeta) = (-1)^n \frac{2^{n+1} n!}{(2n+1)!} H_n^\sigma\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right) u'.$$

This result, however, as has been already pointed out, cannot be accepted unless $\sqrt{(\xi^2 + \eta^2 + \zeta^2)}$ is greater than the longest semi-axis. It also follows from an

investigation, to which I shall now proceed, undertaken with the object of establishing a physical interpretation of ellipsoidal harmonics from the Theory of Attractions similar to that given by CLERK MAXWELL in the case of the sphere.

Physical Interpretation of an Ellipsoidal Harmonic.

17. We consider here the potentials and derivates therefrom of ellipsoids whose layers of equal density are similar and similarly situated to the bounding surface.

Let the density at any internal point fgh be of the form $n\left(1-\frac{f^2}{a^2}-\frac{g^2}{b^2}-\frac{h^2}{c^2}\right)^{n-1}$, then it is shown in THOMSON and TAIT'S 'Natural Philosophy,' vol. 1, part II., p. 526, that the potential at any external point $\xi\eta\zeta$ or $\epsilon\upsilon\upsilon'$ is

$$\pi abc\int_{\epsilon}^{\infty}\left(1-\frac{\xi^2}{a^2+\lambda}-\frac{\eta^2}{b^2+\lambda}-\frac{\zeta^2}{c^2+\lambda}\right)^n\frac{d\lambda}{\mathrm{L}}\,.\qquad\qquad(1).$$

I shall denote this potential by $\pi abc\mathrm{V}_n$.

Suppose now an ellipsoid of any law of internal density ρ, and let U be the potential at an external point $\xi\eta\zeta$; then if we superpose upon this ellipsoid another of density $-\rho$ with its axes in the same directions, and its centre at the point $-\delta\phi_1$, 0, 0, we shall have an outside potential $-\frac{\partial\mathrm{U}}{\partial\xi}\delta\phi_1$ and the density at any internal point fgh will be $-\frac{\partial\rho}{\partial f}\delta\phi_1$. At the surface, however, we shall have a shell the volume density of which at any point $f_1g_1h_1$ will still be ρ.

I remark here that as the density we are to consider is of the form $n\left(1-\frac{f^2}{a^2}-\frac{g^2}{b^2}-\frac{h^2}{c^2}\right)^{n-1}$ where n is an integer, and, as this vanishes at the surface, it can only give rise, by repeated applications of the above process, to expressions which are not evanescent at the surface by being differentiated at least $n-1$ times. In $n-1$ successive differentiations, therefore, in which the surface shell only is to be considered, we may omit consideration of all terms which arise in the differentiation, except those only in each case where the above vanishing expression for the density loses in its index.

Let us now make a second displacement like the first, parallel to the axis of x, through a distance of $\delta\phi_2$, which, for the sake of clearness, we shall suppose less than $\delta\phi_1$. The outside potential will become $\frac{\partial^2\mathrm{U}}{\partial\xi^2}\delta\phi_1\,\delta\phi_2$, and the inside density will be $\frac{\delta^2\rho}{\delta f^2}\delta\phi_1\,\delta\phi_2$, while at the surface there will be a shell of a somewhat complicated character, the nature of which may be best gathered from the following scheme. After the first displacement the state of things in the neighbourhood of any point of the surface in the positive octant is represented by

$$\text{interior, density} = -\frac{\partial\rho}{\partial f}\,\delta\phi_1 \left|\begin{array}{c} \cdots\cdots\delta\phi_1\cdots\cdots \\ \\ \rho \\ \\ \end{array}\right| \text{exterior of ellipsoid.}$$

After the second displacement,

$$\text{interior, density} = \frac{\partial^2\rho}{df^2}\,\delta\phi_1\,\delta\phi_2 \left|\begin{array}{c} \cdots\delta\phi_2\cdots \\ -\frac{\partial\rho}{\partial f}\delta\phi_1 \\ \\ -\rho-\frac{\partial\rho}{\partial f}\,\delta\phi_2 \end{array}\right|\begin{array}{c} \cdots\delta\phi_1-\delta\phi_2\cdots \\ \\ -\frac{\partial\rho}{\partial f}\,\delta\phi_2 \\ \\ \end{array}\left|\begin{array}{c} \cdots\delta\phi_2\cdots \\ \\ \rho \\ \\ \end{array}\right| \text{exterior.}$$

There are thus three shells which a normal to the surface would perforate before reaching the interior, but it is to be remarked that, according to the explanation given above, the density ρ of the first shell may be disregarded, as it can contribute nothing to the final result. In like manner $-\left(\rho+\frac{\partial\rho}{\partial f}\,\delta\phi_1\right)$ in the third shell, which is equal to the first shell geometrically, can contribute nothing, for this quantity is equal to $-\rho$ at a point, $\delta\phi_1$, further along the direction of the axis of x in the first shell. The practical effect of eliminating the first shell and the part referred to of the third will be to leave this state of things—

$$\text{interior, density} = \frac{\partial^2\rho}{\partial f^2}\,\delta\phi_1\,\delta\phi_2 \left|\begin{array}{c} \cdots\cdots\delta\phi_1\cdots\cdots \\ \\ -\frac{\partial\rho}{\partial f}\,\delta\phi_2 \\ \\ \end{array}\right| \text{exterior of ellipsoid;}$$

that is to say, exactly as after the first displacement, provided we substitute $-\frac{\partial\rho}{\partial f}\,\delta\phi_2$ in place of ρ.

By giving exactly equal displacements parallel to the axis of y, we should find for the outside potential $\frac{\partial^2 U}{\partial\eta^2}\,\delta\phi_1\,\delta\phi_2$, for the internal density $\frac{\partial^2\rho}{\partial f^2}\,\delta\phi_1\,\delta\phi_2$, and a similar system of shells, the effective density of which is $-\frac{\partial\rho}{\partial g_1}\,\delta\phi_2$, and whose breadth as measured parallel to the axis of y is $\delta\phi_1$.

Similar results hold also as regards the axis of z.

With this explanation we may now suppose the three systems multiplied respectively by $a^2/a^2+\theta_1$, $b^2/b^2+\theta_1$, $c^2/c^2+\theta_1$, and superposed.

The following statement may then be made:—

The potential

$$\left(\frac{a^2}{a^2+\theta_1}\frac{\partial^2}{\partial\xi^2}+\frac{b^2}{b^2+\theta_1}\frac{\partial^2}{\partial\eta^2}+\frac{c^2}{c^2+\theta_1}\frac{\partial^2}{\partial\zeta^2}\right)\mathrm{U}\,.\,\delta\phi_1$$

arises from an internal density

$$\left(\frac{a^2}{a^2+\theta_1}\frac{\partial^2}{\partial f^2}+\frac{b^2}{b^2+\theta_1}\frac{\partial^2}{\partial g^2}+\frac{c^2}{c^2+\theta_1}\frac{\partial^2}{\partial h^2}\right)\rho\,.\,\delta\phi_1,$$

and from three shells whose volume densities are respectively

$$-\frac{a^2}{a^2+\theta_1}\frac{\partial\rho}{\partial f_1},\ -\frac{b^2}{b^2+\theta_1}\frac{\partial\rho}{dg_1},\ -\frac{c^2}{c^2+\theta_1}\frac{\partial\rho}{\partial h_1}.$$

18. If we make a series of successive operations of the same kind, we shall arrive at this result:—The potential

$$(-1)^m\,\theta_1\ldots\theta_m\,\mathrm{E}_1{}^2\ldots\mathrm{E}_m{}^2\,\mathrm{U}\,.\,\delta\phi_1 \quad (\text{see § 2}) \quad \ldots\ldots \quad (2)$$

arises from internal density

$$\mathfrak{E}_1{}^2\ldots\mathfrak{E}_m{}^2\,\rho\,.\,\delta\phi_1 \quad (\text{see § 2}) \quad \ldots\ldots \quad (3)$$

and from three shells of densities

$$-\frac{a^2}{a^2+\theta_1}\,\mathfrak{E}_2{}^2\ldots\mathfrak{E}_m{}^2\,\frac{\partial\rho}{\partial f_1} \quad \ldots\ldots \quad (4)$$

$$-\frac{b^2}{b^2+\theta_1}\,\mathfrak{E}_2{}^2\ldots\mathfrak{E}_m{}^2\,\frac{\partial\rho}{\partial g_1} \quad \ldots\ldots \quad (5)$$

$$-\frac{c^2}{c^2+\theta_1}\,\mathfrak{E}_2{}^2\ldots\mathfrak{E}_m{}^2\,\frac{\partial\rho}{\partial h_1} \quad \ldots\ldots \quad (6)$$

19. Let us now make the supposition that, in the expression for the law of density in § 17, n is even and equal to $2m$, then we shall show that the expression (3) of § 18 vanishes, provided $\theta_1 \ldots \theta_m$ are the characteristics of an ellipsoidal harmonic $\Theta_1 \ldots \Theta_m$.

The vanishing of (3) is obviously the same thing as the vanishing of

$$\mathrm{E}_1{}^2\ldots\mathrm{E}_m{}^2\,(x^2+y^2+z^2-1)^{2m-1}$$

for the same values of θ.

To prove that this vanishes, I shall avail myself of a theorem given by Dr. FERRERS in a paper on a kindred subject in the 'Quarterly Journal of Mathematics' (vol. 14,

p. 8). With the notation of this paper, the theorem in question may be stated as follows :—

$$“(\lambda x + \mu y + \nu z)^{n-1} = A_{n-1}\left(\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y} + \nu \frac{\partial}{\partial z}\right)^{n-1}(x^2 + y^2 + z^2 - 1)^{n-1}$$

$$+ A_{n-2}(\lambda^2 + \mu^2 + \nu^2)\left(\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y} + \nu \frac{\partial}{\partial z}\right)^{n-3}(x^2 + y^2 + z^2 - 1)^{n-2}$$

$$+ \&c.,$$

where A_{n-1}, A_{n-2}, . . . are numerical multipliers.”

If we differentiate this again by means of the operator

$$\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y} + \nu \frac{\partial}{\partial z}$$

we find

$$(n-1)(\lambda^2 + \mu^2 + \nu^2)(\lambda x + \mu y + \nu z)^{n-2}$$
$$= A_{n-1}\left(\lambda \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial y} + \nu \frac{\partial}{\partial z}\right)^{n}(x^2 + y^2 + z^2 - 1)^{n-1} + \&c.$$

Now this result is true for all possible values of the ratios $\lambda : \mu : \nu$. The coefficients of the various powers and products of λ, μ, ν, when all the expansions are made and the terms collected, must therefore be separately zero. We may accordingly substitute in place of λ, μ, ν differential operators $\frac{\partial}{\partial f}$, $\frac{\partial}{\partial g}$, $\frac{\partial}{\partial h}$ upon a function M (f, g, h), which satisfies LAPLACE's equation.

When this is done, we obtain

$$\left(\frac{\partial}{\partial f}\frac{\partial}{\partial x} + \frac{\partial}{\partial g}\frac{\partial}{\partial y} + \frac{\partial}{\partial h}\frac{\partial}{dz}\right)^{n} \mathrm{M}(f, g, h)(x^2 + y^2 + z^2 - 1)^{n-1} = 0.$$

In this result M is quite arbitrary, subject only to the foregoing condition. We may, therefore, put

$$\mathrm{M}(f, g, h) = \left(\frac{f^2}{a^2 + \theta_1} + \frac{g^2}{b^2 + \theta_1} + \frac{h^2}{c^2 + \theta_1}\right) \ldots \left(\frac{f^2}{a^2 + \theta_m} + \frac{g^2}{b^2 + \theta_m} + \frac{h^2}{c^2 + \theta_m}\right).$$

It is easy then to show that the result finally reduces to

$$E_1^2 \ldots E_m^2 (x^2 + y^2 + z^2 - 1)^{n-1} = 0.$$

Hence (3) of § 18 vanishes.

20. We resume the consideration of the three shells, the expressions for the volume densities of which are given in § 18, (4), (5), and (6). If we bear in mind the remark

contained in § 17 as to what is to be omitted in successive differentiations, the process of reducing the expressions for the densities is easy. Take, for example, $E_2^2\rho$; we have to find

$$\left(\frac{a^2}{a^2+\theta_2}\frac{\partial^2}{\partial f_1^2}+\frac{b^2}{b^2+\theta_2}\frac{\partial^2}{\partial g_1^2}+\frac{c^2}{c^2+\theta_2}\frac{\partial^2}{\partial h_1^2}\right)n\left(1-\frac{f_1^2}{a^2}-\frac{g_1^2}{b^2}-\frac{h_1^2}{c^2}\right)^{n-1}.$$

This, with the omissions alluded to, becomes

$$2^2n(n-1)(n-2)\left\{\frac{f_1^2}{a^2(a^2+\theta_2)}+\text{two similar terms}\right\}\left\{1-\frac{f_1^2}{a^2}-\text{two similar terms}\right\}^{n-3}.$$

If we continue this process for (4) we shall arrive at

$$2^{2m-1}n!\,(-1)^{m+1}\frac{\Theta_2\ldots\Theta_m(f_1,g_1,h_1)}{\theta_2\ldots\theta_m}\frac{f_1}{a^2+\theta_1}.$$

This represents the volume density of the shell indicated by the operation (4), and since, according to the process explained in § 17, the whole of the matter of this shell is included within the boundaries of what is practically the first shell of the series in § 17 the thickness of which at $f_1g_1h_1$ was $\frac{p_1f_1}{a^2}\,\delta\phi_1$, where p_1 is the perpendicular from the centre on the tangent plane at $f_1g_1h_1$, the surface density for (4) at this point is

$$2^{2m-1}(2m)!\,(-1)^{m+1}\frac{\Theta_2\ldots\Theta_m}{\theta_2\ldots\theta_m}\frac{p_1f_1^2}{a^2(a^2+\theta_1}\,\delta\phi_1.$$

A similar treatment applying to the reductions of (5) and (6), we may now superpose (4), (5), and (6), and, observing that $U=\pi abc\,V_{2m}$, present the result in the following form:—

The potential

$$E_1^2\ldots E_m^2\,V_{2m}$$

is due to a shell of surface density

$$2^{2m}(2m)!\,\frac{p_1}{2\pi abc}\,\frac{\Theta_1\ldots\Theta_m(f_1,g_1,h_1)}{\theta_1^2\ldots\theta_m^2}.$$

But this density produces a potential at outside points $\xi\eta\zeta$ or $\epsilon\upsilon\upsilon'$

$$=2^{2m}(2m)!\,\Theta_1\ldots\Theta_m(\xi,\eta,\zeta)\int_\epsilon^\infty\frac{d\lambda}{(\lambda-\theta_1)^2\ldots\mathrm{L}}$$

and at internal points the same expression with ϵ put $=0$.

In the notation described in § 4, this is

$$\mathfrak{G}_{2m}^{\sigma}(\xi, \eta, \zeta) \text{ or } G_{2m}^{\sigma} I_{2m}^{\sigma}(\xi, \eta, \zeta) = \frac{1}{2^{2m}(2m)!} H_{2m}^{\sigma}\left(\frac{\partial}{\partial\xi}, \frac{\partial}{d\eta}, \frac{\partial}{d\zeta}\right) V_{2m}.$$

The proof has been established for one particular type of harmonic of even degree, but it is almost identical for every variety of case included in the scheme of § 3. The general result is

$$\mathfrak{G}_n^{\sigma} = (-1)^n \frac{1}{2^n n!} H_n^{\sigma}\left(\frac{\partial}{\partial\xi}, \frac{\partial}{d\eta}, \frac{\partial}{d\zeta}\right) V_n.$$

When $n = 0$, which is an exceptional case, we may take

$$\mathfrak{G}_0 = I_0 = \int_{\epsilon}^{\infty} \frac{d\lambda}{L}.$$

21. The expansion for the potential due to a simple shell, whose semi-axes are a, b, c, at a sufficiently distant external point $\xi\eta\zeta$, is easily shown to be

$$\text{mass of shell}\left(1 + \frac{D^2}{3!} + \ldots + \frac{D^{2m}}{(2m+1)!} + \ldots\right)u,$$

where

$$u^{-2} = \xi^2 + \eta^2 + \zeta^2.$$

Hence, the potential due to any one of the simple shells into which the ellipsoid may be divided, with the law of density of § 17, is

$$4\pi abc\, \theta^2\partial\theta \,.\, n(1-\theta^2)^{n-1}\left(1 + \frac{\theta^2 D^2}{3!} + \ldots\right)u.$$

Integrating this between the limits 0 and 1, and remembering that the result is $\pi abc V_n$ we find

$$\begin{aligned} V_n &= 2\Gamma(n+1) \sum_{m=0}^{m=\infty} \frac{\Gamma(m+\frac{3}{2})}{\Gamma(2m+2)\,\Gamma(m+n+\frac{3}{2})} D^{2m}u \\ &= \frac{2^{2n+1}\, n!\, n!}{(2n+1)!} \sum_{m=0}^{m=\infty} \frac{D^{2m}u}{2^m m!\,(2n+2m+1)(2n+2m-1)\ldots(2n+3)}. \end{aligned}$$

On substituting this expression for V_n, in the last line of § 20, we obtain the same result as in § 16. It may further be remarked that V_n may be exhibited in the form

$$\int_0^1 (1-\theta^2)^n (e^{\theta D} + e^{-\theta D})\, d\theta \,.\, u.$$

Applications.—1. *Induced Magnetism.*

22. To show what calculations are required in solving physical problems involving boundary conditions at the surface of an ellipsoid, it will be sufficient to take the problem of finding the magnetism induced in a solid ellipsoid under given magnetic forces.

If $\xi\eta\zeta$ be the magnetic centre of any external magnet, and xyz any point inside of the ellipsoid, we may represent the potential at the latter point due to the outside magnet in the form

$$\left(\mathrm{A}\frac{\partial}{\partial\xi} + \mathrm{B}\frac{\partial}{\partial\eta} + \mathrm{C}\frac{\partial}{\partial\zeta}\ldots\right)\frac{1}{r},$$

where r is the distance between $\xi\eta\zeta$ and xyz (CLERK MAXWELL's 'Electricity and Magnetism,' vol. 2, § 391). Hence, taking the expansion of $1/r$, given in § 14 above, we see that for any known system of magnets we may always express the potential due to them in the form of a series, the general term of which is $\alpha_n{}^\sigma \mathrm{G}_n{}^\sigma(xyz)$ where $\alpha_n{}^\sigma$ is a known constant.

23. If we assume corresponding series for the potential due to the induced magnetism it will be necessary, in order to satisfy the condition of continuity of the magnetic induction at the surface, to find thereat the rates of normal variation of $\mathrm{G}_n{}^\sigma(x, y, z)$ and $\mathfrak{G}_n{}^\sigma(x, y, z)$. Take first

$$\frac{\partial \mathrm{G}_n{}^\sigma}{\partial n} = p\left(\frac{x}{a^2}\frac{\partial}{\partial x} + \frac{y}{b^2}\frac{\partial}{\partial y} + \frac{z}{c^2}\frac{\partial}{\partial z}\right)\mathrm{G}_n{}^\sigma,$$

and observe that $\mathrm{G}_n{}^\sigma$ will contain a series of factors Θ, upon any one of which the operator within brackets gives at the surface $-2\Theta/\theta$. We may therefore write

$$\frac{\partial \mathrm{G}_n{}^\sigma}{\partial n} = l_n{}^\sigma \, . \, p\mathrm{G}_n{}^\sigma,$$

where

$$l_n{}^\sigma = -2\Sigma\frac{1}{\theta} + \frac{p}{a^2} + \frac{q}{b^2} + \frac{r}{c^2};$$

the quantities p, q, r being either unity or zero, according as $\mathrm{G}_n{}^\sigma$ does or does not contain x, y, z as factors.

In like manner, if Π denote the product of all the values of θ, as well as a, b, c corresponding to factors x, y, z, we shall have at the surface

$$\frac{\partial}{\partial n}\mathfrak{G}_n{}^\sigma = \frac{\partial}{\partial n}(\mathrm{G}_n{}^\sigma \mathrm{I}_n{}^\sigma) = \left(l_n{}^\sigma \mathrm{I}_n{}^\sigma(0) - \frac{2}{abc\,\Pi^2}\right)p\mathrm{G}_n{}^\sigma.$$

24. The inside and outside potentials due to the induced magnetism may readily be found from the usual conditions in the following forms, viz.: $A_n^\sigma G_n^\sigma I_n^\sigma(0)$ and $A_n^\sigma \mathfrak{G}_n^\sigma$, where

$$\frac{4\pi\kappa\alpha_n^\sigma}{A_n^\sigma} + 4\pi\kappa I_n^\sigma(0) + \frac{2}{abc\,\Pi^2 l_n^\sigma} = 0,$$

κ being the magnetic susceptibility of the ellipsoid.

The potential due to the magnetism induced in a solid sphere admits of a compact expression. If V denote the potential due to the inducing system, then at any point in the interior of the sphere the potential due to the induced magnetism is

$$-\frac{2\pi\kappa}{1+2\pi\kappa}r^{-(2+4\pi\kappa)^{-1}}\int_0^r r^{(2+4\pi\kappa)^{-1}}\frac{\partial V}{\partial r}\,dr.$$

The outside potential may be expressed in a similar form.

Applications.—2. Potential due to a thin shell bounded by similar and similarly situated ellipsoids, the density of which varies inversely as the cube of the distance from a fixed point.

25. First let the fixed point $\xi\eta\zeta$ or $\epsilon\upsilon\upsilon'$ be outside of the ellipsoid, and let r denote the distance from it to a point fgh on the surface: consider then the integral

$$\iint \frac{G_n^\sigma(f,g,h)}{r}\,p\,dS = 2\pi abc\,\{\Pi(\theta)\}^2 G_n^\sigma I_n^\sigma(\xi,\eta,\zeta).$$

If we operate on both sides of this relation by means of the operator

$$\frac{\xi}{a^2}\frac{\partial}{\partial\xi} + \frac{\eta}{b^2}\frac{\partial}{\partial\eta} + \frac{\zeta}{c^2}\frac{\partial}{\partial\zeta},$$

we may arrange the result of the left-hand operation thus:

$$\iint \frac{G_n^\sigma(f,g,h)}{r^3}\left\{\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)-\left(\frac{f(f-\xi)}{a^2}+\frac{g(g-\eta)}{b^2}+\frac{h(h-\zeta)}{c^2}\right)\right\}p\;dS,$$

which, if we put $1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}=u$, is the same as

$$u\iint\frac{G_n^\sigma p\,dS}{r^3} + \iint G_n^\sigma\frac{\partial}{\partial n}\frac{1}{r}\,.\,dS.$$

If we now take the expansion $1/r$ given in § 14, and, observing that at the surface,

$\frac{\partial}{\partial n} \mathrm{G}_m{}^\sigma = l_m{}^\sigma \,.\, p\mathrm{G}_m{}^\sigma$ (§ 23), enter it in the second integral, it will be seen that all the terms but one disappear, and we find

$$\iint \mathrm{G}_n{}^\sigma \frac{\partial}{\partial n} \frac{1}{r}\, d\mathrm{S} = 2\pi \mathrm{G}_n{}^\sigma \mathrm{I}_n{}^\sigma (\xi, \eta, \zeta)\, l_n{}^\sigma abc \{\Pi(\theta)\}^2.$$

On the right-hand side we have to find

$$2\pi abc \{\Pi(\theta)\}^2 \left(\frac{\xi}{a^2}\frac{\partial}{\partial \xi} + \frac{\eta}{b^2}\frac{\partial}{\partial \eta} + \frac{\zeta}{c^2}\frac{\partial}{\partial \zeta}\right) \mathrm{G}_n{}^\sigma \mathrm{I}_n{}^\sigma (\xi, \eta, \zeta).$$

Suppose Θ one of the factors of $\mathrm{G}_n{}^\sigma (\xi, \eta, \zeta)$, the operator upon Θ alone would give

$$\frac{2\xi^2}{a^2(a^2+\theta)} + \frac{2\eta^2}{b^2(b^2+\theta)} + \frac{2\zeta^2}{c^2(c^2+\theta)} \quad \text{or} \quad -\frac{2}{\theta}(\Theta + u).$$

Hence,

$$\left(\frac{\xi}{a^2}\frac{\partial}{\partial \xi} + \ldots\right) \mathrm{G}_n{}^\sigma (\xi, \eta, \zeta) = \left(l_n{}^\sigma - 2u\,\Sigma \frac{1}{\theta\Theta}\right) \mathrm{G}_n{}^\sigma (\xi, \eta, \zeta),$$

and

$$\left(\frac{\xi}{a^2}\frac{\partial}{\partial \xi} + \ldots\right) \mathrm{I}_n{}^\sigma (\xi, \eta, \zeta) = -\frac{1}{\{\Pi(\theta-\epsilon)\}^2 \sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}} \left(\frac{\xi}{a^2}\frac{d\epsilon}{d\xi} + \ldots\right)$$

$$= \frac{2\varpi^2 u}{\{\Pi(\theta-\epsilon)\}^2 \epsilon \sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}},$$

ϖ being the perpendicular from the centre on the tangent plane to the confocal ellipsoid at $\xi\eta\zeta$.

Performing the complete operation on the right-hand side, and comparing the result with that on the left, the term containing $l_n{}^\sigma$ cancels, and we obtain

$$\iint \frac{\mathrm{G}_n{}^\sigma p\, d\mathrm{S}}{r^3} = \frac{4\pi\varpi^2 abc \{\Pi(\theta)\}^2 \mathrm{G}_n{}^\sigma (\xi, \eta, \zeta)}{\{\Pi(\theta-\epsilon)\}^2 \epsilon \sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}} - 4\pi abc \{\Pi(\theta)\}^2 u \Sigma \frac{1}{\theta\Theta} \mathrm{G}_n{}^\sigma \mathrm{I}_n{}^\sigma (\xi, \eta, \zeta).$$

Now, by § 10, if r on the left-hand side denote the distance between $\xi\eta\zeta$ and fgh and, on the right, between $\xi\eta\zeta$ and any element $d\mathrm{S}$ on the surface, then

$$\frac{1}{r^3} = \frac{1}{4\pi abc} \iint \frac{p\, d\mathrm{S}}{r^3} + \ldots + \Sigma \frac{\mathrm{G}_n{}^\sigma (f, g, h) \iint \frac{\mathrm{G}_n{}^\sigma}{r^3} p\, d\mathrm{S}}{\iint (\mathrm{G}_n{}^\sigma)^2 p\, d\mathrm{S}} + \ldots$$

Hence, on entering the values of the integrals just found,

$$\frac{1}{r^3} = \frac{\varpi^2}{\epsilon\sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}}\left\{1 + \ldots + 4\pi abc\left(\frac{\Pi(\theta)}{\Pi(\theta-\epsilon)}\right)^2 \frac{G_n^\sigma(f,g,h)\,G_n^\sigma(\zeta,\eta,\zeta)}{\iint (G_n^\sigma)^2\, p\, dS} + \ldots\right\}$$

$$- 4\pi \sum_{n=2}^{n=\infty} \frac{G_n^\sigma(f,g,h)\,\Sigma\,\dfrac{1}{\theta\Theta}\,G_n^\sigma I_n^\sigma(\xi,\eta,\zeta)}{\iint (H_n^\sigma)^2\, ds}\,.$$

If we multiply both sides of the last found relation by p, the perpendicular from the centre to the tangent plane at fgh, we may find the potentials outside and inside the shell due to density p/r^3, for we know the potentials which $p\,G_n^\sigma(f, g, h)$ will produce.

It will be observed that the second of the two series expressing $1/r^3$ begins with harmonics of the second degree, and contains those harmonics only which have at least one factor of the form Θ. From this circumstance, by taking the first four terms of the outside potential, viz., as far as harmonics of the first degree inclusive, it may be shown that, to the same degree of approximation, the potential at any outside point x, y, z is the same as if the mass were collected at a certain internal point.* If we denote by I_0, xI_1', yI_1'', zI_1''' the first four harmonics at an external point xyz, then the potential at this point is

$$\frac{2\pi abc\varpi^2}{\epsilon\sqrt{\{(a^2+\epsilon)(b^2+\epsilon)(c^2+\epsilon)\}}}\left\{I_0 + \frac{3a^2\xi}{a^2+\epsilon}\frac{xI_1'}{a^2} + \frac{3b^2\eta}{b^2+\epsilon}\frac{yI_1''}{b^2} + \frac{3c^2\zeta}{c^2+\epsilon}\frac{zI_1'''}{c^2}\right\}.$$

But, by § 14, this to the same degree of approximation is the same as if the mass were collected at the point $\alpha\beta\gamma$, such that

$$\frac{\alpha}{a^2} = \frac{\xi}{a^2+\epsilon}, \qquad \frac{\beta}{b^2} = \frac{\eta}{b^2+\epsilon}, \qquad \frac{\gamma}{c^2} = \frac{\zeta}{c^2+\epsilon}\,.$$

26. It may be pointed out that the two points $\xi\eta\zeta$ and $\alpha\beta\gamma$ possess a remarkable geometrical property which may be stated as follows :—

If xyz be any point on the surface of the ellipsoid, then

$$(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$$
$$+ \epsilon\left\{\left(\frac{x-\alpha}{a}\right)^2 + \left(\frac{y-\beta}{b}\right)^2 + \left(\frac{z-\gamma}{c}\right)^2\right\}.$$

By means of this geometrical theorem it is easy to demonstrate the property proved above, that $\alpha\beta\gamma$ is the centre of inertia of the shell just discussed. The geometrical theorem occurs, though in a different form, in a paper by CAYLEY in the 'Proceedings of the London Mathematical Society' (vol. 6, p. 58).

* The property referred to was set as a problem by Mr. TEMPERLEY in the Mathematical Tripos, 1880.

27. If the point $\xi\eta\zeta$ be inside of the shell, the foregoing expressions and series will take different forms.

The proper expansion of $1/r$ will then be

$$\tfrac{1}{2}\mathrm{I}_0(f, g, h) + \ldots + 2\pi\Sigma\frac{\mathrm{G}_n{}^\sigma(\xi, \eta, \zeta)\,\mathrm{G}_n{}^\sigma\,\mathrm{I}_n{}^\sigma(f, g, h)}{\iint(\mathrm{H}_n{}^\sigma)^2\,ds} + \ldots$$

and $\dfrac{\partial}{\partial n}\dfrac{1}{r}$ will be

$$2\pi\Sigma\frac{\mathrm{G}_n{}^\sigma(\xi\eta\zeta)}{\iint(\mathrm{H}_n{}^\sigma)^2\,ds}\left(l_n{}^\sigma\,\mathrm{I}_n{}^\sigma(0) - \frac{2}{abc\{\Pi(\theta)\}^2}\right)p\,\mathrm{G}_n{}^\sigma(f, g, h).$$

The left-hand side in this case will, therefore, be

$$u\iint\frac{\mathrm{G}_n{}^\sigma p\,d\mathrm{S}}{r^3} + 2\pi\,\mathrm{G}_n{}^\sigma(\xi\eta\zeta)\left[abc\,\{\Pi(\theta)\}^2\,l_n{}^\sigma\,\mathrm{I}_n{}^\sigma(0) - 2\right].$$

The right-hand side will be

$$2\pi abc\,\{\Pi(\theta)\}^2\left(l_n{}^\sigma - 2u\,\Sigma\frac{1}{\theta\Theta}\right)\mathrm{G}_n{}^\sigma(\xi\eta\zeta)\,\mathrm{I}_n{}^\sigma(0).$$

Hence

$$u\iint\frac{\mathrm{G}_n{}^\sigma}{r^3}p\,d\mathrm{S} = 4\pi\mathrm{G}_n{}^\sigma(\xi\eta\zeta) - 4\pi abc\,\{\Pi(\theta)\}^2\,u\,\Sigma\frac{1}{\theta\Theta}\,\mathrm{G}_n{}^\sigma(\xi\eta\zeta)\,\mathrm{I}_n{}^\sigma(0),$$

and

$$\frac{u}{r^3} = \frac{1}{abc} + \ldots + 4\pi\frac{\mathrm{G}_n{}^\sigma(fgh)\,\mathrm{G}_n{}^\sigma(\xi\eta\zeta)}{\iint(\mathrm{G}_n{}^\sigma)^2\,pd\mathrm{S}} - 4\pi u\sum_{n=2}^{n=\infty}\frac{\mathrm{G}_n{}^\sigma(fgh)\,\mathrm{I}_n{}^\sigma(0)\,\Sigma\dfrac{1}{\theta\Theta}\,\mathrm{G}_n{}^\sigma(\xi\eta\zeta)}{\iint(\mathrm{H}_n{}^\sigma)^2\,ds}.$$

The centre of inertia of this shell is the point $\xi\eta\zeta$ itself; and its mass is clearly given by a different expression to what it was in the former case.

If unit of electricity be placed at an internal point $\xi\eta\zeta$ and σ denote the density at fgh, then $-4\pi\sigma$ is equal to the first of the two last series multiplied by p. The above result, therefore, shows how far $4\pi\sigma$ is different to $-pu/r^3$, which corresponds to the similar expression for the sphere.

Applications.—3. Electrical Capacity.

28. The theorem which forms the subject of this article is not a special illustration of the use of ellipsoidal harmonics, except in so far that GREEN's function can be determined for an ellipsoid in terms of those functions. I shall now show that the determination of GREEN's function for any surface suffices also to determine in a simple form the capacity of the inverted surface.

If R be the radius of inversion, r the radius vector from the electrified point to an element $\delta\mathrm{S}$ of the conductor, at which the density is σ; r', $\delta\mathrm{S}'$, σ' similar quantities for the inverted surface: then, by well-known results,

$$\sigma' \, \delta S' = \left(\frac{R}{r'}\right)^3 \sigma \left(\frac{r'}{r}\right)^2 \delta S = \frac{R}{r} \sigma \, . \, \delta S.$$

$$\therefore \quad \iint \sigma' \, dS' = R \iint \frac{\sigma \, dS}{r} \, .$$

On the left-hand side, the integrations cover the whole of the inverted surface, and on the right the whole of the original surface. We have supposed the electrified point to be unity. If we take it equal to R, then the potential of the freely electrified inverted surface will be unity, and the above result shows that its capacity is $R^2 u$ where $-u$ is the potential at the electrified point, of unit magnitude, due to the charge induced by it in the original conductor. For instance, let unit of electricity be placed at a distance f from the centre of a spherical conductor of radius a, and let the sphere be inverted into itself. Then

$$R^2 = f^2 - a^2 \qquad \text{and} \qquad u = \frac{a}{f} \Big/ \left(f - \frac{a^2}{f}\right).$$

Therefore $R^2 u = a$, as it ought to be.

29. It may be pointed out that the foregoing theorem is suitable for the determination of the capacity of the inverted surfaces derived from

$$V \equiv \frac{1}{r} - \frac{a}{f} \frac{1 + e \tan^n \frac{1}{2} \theta \cos n\phi}{r'} = 0,$$

where n is integral, r, r' are the radii vectores from fixed points A and B to any point P; PBA $= \theta$; and the longitudinal angle of the plane PBA $= \phi$. We shall suppose n even and $f > a(1 + e)$, the distance AB being, as before, $f - a^2/f$. Then so long as the expression

$$1 + e \tan^n \tfrac{1}{2} \theta \cos n\phi$$

is negative, it is impossible to find points for which $V = 0$. If, however, it is positive, there are real points until it reaches the value f/a, when r and r' become infinite. Hence, when $\cos n\phi$ is positive, the equation

$$1 + e \tan^n \tfrac{1}{2} \theta \cos n\phi = f/a$$

is that of a cone parallel to the conical asymptote to the surface $V = 0$, the vertex of which is A. The inverted surface, with respect to A, will therefore have a conical protuberance, terminating at the point A, with n corrugations symmetrically ranged upon it.

If the radius of inversion be, as before, $\sqrt{(f^2 - a^2)}$, the capacities of all such inverted surfaces will be a.

Inverted figures obtained from ellipsoids present many varieties according to their degrees of prolateness or oblateness, and the position of the point of inversion. The expression for GREEN'S function in the general case of the ellipsoid with three unequal axes is, however, of too complicated a character to admit of interesting results.

Ellipsoids of Revolution.

Reduction of the K-*Functions.*

30. We here consider the case in which two axes of the ellipsoid are equal, say $a = b$.

It will be convenient to take one particular form for discussion; let us, therefore, choose $\Theta_1 \ldots \Theta_n$, and find to what the corresponding function $K_1 \ldots K_n$ reduces when all the characteristics $= -a^2$.

In the equations (3) § 3 write $b^2 + \theta_1 = (a^2 - b^2)(-q_1)$, $a^2 + \theta_1 = (a^2 - b^2)(1 - q_1)$, and similarly for $\theta_2 \ldots \theta_n$; multiply by $a^2 - b^2$ throughout, and put $a = b$. We obtain

$$\frac{1}{q_1 - 1} + \frac{1}{q_1} + \frac{4}{q_1 - q_2} + \&c. = 0, \&c.$$

Hence all the roots satisfy an equation $f(q) = 0$ such that

$$\frac{1}{q-1} + \frac{1}{q} + 2\frac{d^2f}{dq^2} \Big/ \frac{df}{dq} = 0.$$

This equation being of the same degree as f itself, we must have, identically,

$$q(q-1)\frac{d^2f}{dq^2} + \tfrac{1}{2}(2q-1)\frac{df}{dq} = n^2f,$$

or

$$\sqrt{\{q(1-q)\}}\frac{d}{dq}\sqrt{\{q(1-q)\}}\frac{df}{dq} + n^2f = 0,$$

or

$$\frac{d^2f}{d\chi^2} + 4n^2f = 0,$$

where $q = \sin^2\chi$.

Hence $f = \cos 2n\chi$ or $\sin 2n\chi$, the former value being appropriate to the harmonic under consideration, the latter to the form $xy\,\Theta_1 \ldots \Theta_n$ which leads by transformation (§ 6) to the same differential equation.

Since any factor K in the case we have supposed is proportional to

$$x^2(-q) + y^2(1-q),$$

i.e., to

$$(x^2 + y^2)\left(\frac{y^2}{x^2 + y^2} - q\right) \text{ or } (x^2 + y^2)(\sin^2\phi - q)$$

if we disregard unessential constants, we have $K_1 \ldots K_n$ proportional to

$$(x^2 + y^2)^n(\sin^2\phi - \sin^2\chi_1)\ldots(\sin^2\phi - \sin^2\chi_n)$$

where $\chi_1 \ldots \chi_n$ satisfy $\cos 2n\chi = 0$.

The harmonic, therefore, becomes proportional to

$$(x^2 + y^2)^n \cos 2n\phi,$$

or, as it is usual to write it, $\frac{1}{2}(\xi^{2n} + \eta^{2n})$, where $\xi = x + jy$, $\eta = x - jy$.

It is noteworthy that, in this particular case, $\Theta_1 \ldots \Theta_n$ reduces to the same result.

31. Next let $\Theta_1 \ldots \Theta_n$ have σ values equal to $-a^2$ and $n - \sigma$ between $-a^2$ and $-c^2$.

It will be found that the equation to determine the equal roots is identically the same as in the preceding article, except that σ takes the place of n. One factor of $K_1 \ldots K_n$ will therefore be $\cos 2\sigma\phi$.

Let each of the remaining $n - \sigma$ roots satisfy an equation $f(\theta) = 0$: then, by the equations (3), § 3, each of the roots in question must satisfy

$$\frac{4\sigma + 2}{a^2 + \theta} + \frac{1}{c^2 + \theta} + 2\,\frac{d^2f}{d\theta^2}\Big/\frac{df}{d\theta} = 0.$$

Write herein $c^2 + \theta = -(a^2 - c^2)\,p$, $a^2 + \theta = (a^2 - c^2)(1 - p)$; then

$$-2p(1 - p)\frac{d^2f}{dp^2} + \{(4\sigma + 3)p - 1\}\frac{df}{dp} = 0,$$

and this being of the same degree in p as f itself the left-hand side must be identically the same as $(n - \sigma)(2n + 2\sigma + 1)f$.

Now put $p = \mu^2$; then the equation to determine the form of f is

$$(1 - \mu^2)\frac{d^2f}{d\mu^2} - 2(2\sigma + 1)\mu\frac{df}{d\mu} + (2n - 2\sigma)(2n + 2\sigma + 1)f = 0.$$

But this equation (see FERRERS, IV., § 9) is satisfied by

$$f = (1 - \mu^2)^\sigma \frac{d^{2\sigma}}{d\mu} \cdot P_{2n}(\mu).$$

If, now, we omit unessential constants, a K-factor becomes $z^2 - (x^2 + y^2 + z^2)\,p$. Hence the $n - \sigma$ factors produce, when multiplied together,

$$(x^2 + y^2 + z^2)^{n-\sigma}\frac{d^{2\sigma}}{d\mu}\,P_{2n}(\mu).$$

The complete K-spherical harmonic is therefore

$$(\xi^{2\sigma} + \eta^{2\sigma})(x^2 + y^2 + z^2)^{n-\sigma}\frac{d^{2\sigma}}{d\mu}\,P_{2n}(\mu).$$

32. We have thus investigated one particular case of the scheme (1), § 3, but all the other cases may be similarly treated. It will be found that functions of the forms $(1, z, x, xz)\, K_1 \ldots K_n$ have $\xi^\sigma + \eta^\sigma$ as their longitudinal factor, σ being even for the first two forms and odd for the second two, and that $(y, yz, xy, xyz)\, K_1 \ldots K_n$ have $\xi^\sigma - \eta^\sigma$, σ being odd for the first two and even for the second two.

The general form to which the K-harmonic of n^{th} degree reduces is

$$r^n \begin{matrix}\cos\sigma\phi\\ \sin\sigma\phi\end{matrix} (1-\mu^2)^{\sigma/2} \frac{d^\sigma}{d\mu} \mathrm{P}_n(\mu),$$

and is seen to be that of the ordinary tesseral and sectorial system.

33. Reverting to the particular type $\Theta_1 \ldots \Theta_n$ we may arrange the solutions as follows :—First with the n values of θ each $= -a^2$; next with $n-1$ values $= -a^2$ and one between $-a^2$ and $-c^2$; and so on; the total number of values, $= -a^2$, used in the process, being $n + (n-1) + \ldots + 1 = \frac{1}{2} n(n+1)$, *i.e.*, half the whole number of the values of θ (§ 5). This shows in what manner the values of θ are distributed in the case of the ellipsoid of three unequal axes. A harmonic of the type in question may, in the general case, have r values of θ between $-a^2$ and $-b^2$ and $n-r$ between $-b^2$ and $-c^2$, where r is any integer from 0 up to n.

Internal Harmonics of a Prolate Spheroid.

34. It is desirable at this stage to effect an alteration of meaning in two of the symbols, viz., $\mathrm{H}_n{}^\sigma$ and $\mathrm{G}_n{}^\sigma$, as used to express corresponding internal spherical and spheroidal harmonics. It will now be convenient to adopt the spherical harmonic system in THOMSON and TAIT'S 'Natural Philosophy,' and CLERK MAXWELL'S 'Electricity and Magnetism,' and to express corresponding spheroidal harmonics in conformity therewith. The new meaning of $\mathrm{H}_n{}^\sigma$ may be taken as expressed by any one of the following well-known forms, collected together here for future reference :—

i. $$\frac{(n+\sigma)!}{2^{2\sigma} n!\,\sigma!} (\xi^\sigma + \eta^\sigma) \left\{ z^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{4(\sigma+1)} z^{n-\sigma+2} (x^2+y^2) + \ldots \right\}.$$

ii. $$\frac{(2n)!}{2^{n+\sigma} n!\, n!} (\xi^\sigma + \eta^\sigma) \left\{ z^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{2(2n-1)} z^{n-\sigma+2} (x^2+y^2+z^2) + \ldots \right\}.$$

iii. $$\frac{(n+\sigma)!\,\sigma!}{(2\sigma)!\,n!} (\xi^\sigma + \eta^\sigma) \frac{1}{\pi} \int_0^\pi (z + j\rho\cos\theta)^{n-\sigma} \sin^{2\sigma}\theta\, d\theta.$$

iv. $$2^{-\sigma}(-j)^\sigma \frac{(n+\sigma)!\,(n-\sigma)!}{n!\,n!} 2\cos\sigma\phi \frac{1}{\pi} \int_0^\pi (z + j\rho\cos\theta)^n \cos\sigma\theta\, d\theta.$$

If $\sigma = 0$ these expressions must be divided by 2 to express $r^n \mathrm{P}_n(\mu)$. Should there

be occasion to specially denote the corresponding harmonics when the longitudinal factor is $\xi^\sigma - \eta^\sigma$ in place of $\xi^\sigma + \eta^\sigma$ we shall use the symbol H'^n_σ.

35. According to § 15 the corresponding spheroidal harmonic is given by

$$\begin{aligned} G_n^\sigma &= \left(1 - \frac{c^2 - a^2}{2(2n-1)}\frac{\partial^2}{\partial z^2} + \ldots\right) H_n^\sigma \\ &= H_n^\sigma + AH_{n-2}^\sigma + BH_{n-4}^\sigma + \ldots, \end{aligned}$$

where A, B, . . . are functions of n and σ arising in the process of differentiation.

It will conduce to brevity if we introduce a new symbol, defined by the relation

$$H_n^\sigma = r^n . 2 \cos \sigma\phi \, (1 - \mu^2)^{\sigma/2} \chi_n^\sigma(\mu),$$

so that

$$\begin{aligned} (1 - \mu^2)^{\sigma/2} \chi_n^\sigma(\mu) &= 2^{-\sigma} \frac{(n-\sigma)!}{n!} (1 - \mu^2)^{\sigma/2} \frac{d^\sigma}{d\mu} P_n(\mu) \\ &= 2^{-\sigma} (-j)^\sigma \frac{(n+\sigma)!\,(n-\sigma)!}{n!\; n!} \frac{1}{\pi}\int_0^\pi (\mu + j\nu \cos\theta)^n \cos\sigma\theta \, d\theta. \end{aligned}$$

Taking now the case of a prolate spheroid, *i.e.*, supposing $c > a$, and writing $c^2 - a^2 = \gamma^2$, we see that G_n^σ may be thrown into either of the following forms :—

$$\text{i.} \quad \frac{2^{n+\sigma}(n+\sigma)!\; n!\; \sigma!}{(2n)!\;(2\sigma)!} \gamma^{n-\sigma} (\xi^\sigma + \eta^\sigma) \frac{1}{\pi}\int_0^\pi \chi_n^\sigma\left(\frac{z + j\rho\cos\theta}{\gamma}\right) \sin^{2\sigma}\theta \, d\theta,$$

$$\text{ii.} \quad (-j)^\sigma \frac{2^{n-\sigma}(n+\sigma)!\;(n-\sigma)!}{(2n)!} \gamma^n 2\cos\sigma\phi \frac{1}{\pi}\int_0^\pi P_n\left(\frac{z + j\rho\cos\theta}{\gamma}\right) \cos\sigma\theta \, d\theta,$$

and

$$G_n = \frac{2^n\, n!\; n!}{(2n)!} \gamma^n \frac{1}{\pi}\int_0^\pi P_n\left(\frac{z + j\rho\cos\theta}{\gamma}\right) d\theta.$$

36. The expressions just found enable us to express G_n^σ in terms of ellipsoidal coordinates.

In the formulæ

$$(c^2 - a^2) z^2 = (c^2 + \epsilon)(c^2 + v'), \qquad (c^2 - a^2)\rho^2 = -(a^2 + \epsilon)(a^2 + v')$$

put

$$\begin{aligned} (c^2 - a^2)\mu^2 &= c^2 + \epsilon & \qquad (c^2 - a^2)\nu^2 &= -(a^2 + \epsilon) \\ (c^2 - a^2)\mu_1^2 &= c^2 + v' & \qquad (c^2 - a^2)\nu_1^2 &= -(a^2 + v'). \end{aligned}$$

Then

$$\begin{aligned} P_n\left(\frac{z + j\rho\cos\theta}{\gamma}\right) &= P_n\{\mu\mu_1 + \nu\nu_1\cos(\pi - \theta)\} \\ &= P_n(\mu) P_n(\mu_1) + \ldots \\ &\quad + (-1)^\sigma 2^{2\sigma+1} \frac{n!\; n!}{(n+\sigma)!\;(n-\sigma)!} \nu^\sigma \nu_1^\sigma \chi_n^\sigma(\mu) \chi_n^\sigma(\mu_1) \cos\sigma\theta + \&c. \end{aligned}$$

First expressing $\nu^\sigma \chi_n^\sigma$ in terms of ellipsoidal coordinates according to the formula of the preceding article, and then entering the series just obtained in the second expression for G_n^σ (ii. in § 35), we may exhibit the final result, after reduction, in the form

$$G_n^\sigma = \alpha_n^\sigma . 2 \cos \sigma\phi . f_n^\sigma(\epsilon)\, f_n^\sigma(v'),$$

where

$$\alpha_n^\sigma = (-j)^\sigma\, 2^{n-\sigma} \frac{\{(n+\sigma)!\,(n-\sigma)!\}^2}{(2n)!\; n!\; n!}$$

$$f_n^\sigma(\epsilon) = \frac{1}{\pi}\int_0^\pi \{\sqrt{(c^2+\epsilon)} - \sqrt{(a^2+\epsilon)}\cos\theta\}^n \cos\sigma\theta\, d\theta$$

$$f_n^\sigma(v') = \frac{1}{\pi}\int_0^\pi \{\sqrt{(c^2+v')} - \sqrt{(a^2+v')}\cos\theta\}^n \cos\sigma\theta\, d\theta.$$

External Harmonics of a Prolate Spheroid.

37. Since the form of G_n^σ is completely determined in the foregoing articles, the present part of this investigation may be considered as a method of evaluating the integral I_n^σ.

Two preliminary remarks seem necessary:—(1) The result established in § 20, expressing $\mathfrak{G}_n^\sigma$ or $G_n^\sigma I_n^\sigma$ in terms of a differential operation upon V_n, is true for the altered meanings of H_n^σ and G_n^σ, for the change merely amounts to the removal of a constant from the two sides of an equation. In like manner the various forms of expansion investigated in §§ 10 and 15 are clearly unaffected by the change. (2) Since, by § 15, G_n^σ is determined from H_n^σ by an operator $f(D^2)$ where D^2 may be written $(a^2-c^2)\dfrac{\partial^2}{\partial x^2} + (b^2-c^2)\dfrac{\partial^2}{\partial y^2}$, in which the differences of the squares of the semi-axes only appear, and since I_n^σ and V_n are both independent of any particular confocal, it is clear that we may select that confocal of the system which is the most readily dealt with. This will obviously be the confocal ellipse to which we will suppose the ellipsoid contracted, retaining in its contraction, however, the same law of internal density. When this is done the result may be stated thus:—

$$\frac{\Gamma(n+1)\,\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\iint\left(1-\frac{f^2}{a^2-c^2}-\frac{g^2}{b^2-c^2}\right)^{n-\frac{1}{2}} \frac{df\,dg}{\sqrt{\{(x-f)^2+(y-g)^2+z^2\}}} = \pi\sqrt{\{(a^2-c^2)(b^2-c^2)\}}\, V_n(xyz),$$

V_n retaining the same meaning as before.

If the ellipsoid be a prolate spheroid it ultimately contracts to the line joining the two foci, the length of which is $2\sqrt{(c^2-a^2)}$ or 2γ. We shall then have, by an easy integration,

$$V_n(x, y, z) = \frac{1}{\gamma}\int_{-\gamma}^{\gamma} \frac{\left(1 - \frac{u^2}{\gamma^2}\right)^n}{\sqrt{\{x^2 + y^2 + (z-u)^2\}}}\,du.$$

38. Confining attention, as before, to the case of the prolate spheroid, we shall enter the expression for V_n just found in the relation of § 20, viz.,

$$G_n^\sigma I_n^\sigma(x, y, z) = \frac{(-1)^n}{2^n n!} H_n^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) V_n.$$

The form of H_n^σ to be used is ii. in § 34, and it will be observed that, in making the substitutions for x, y, z in differential operators, $\xi^\sigma + \eta^\sigma$ must be replaced by

$$2^\sigma\left(\frac{\partial^\sigma}{\partial \xi} + \frac{\partial^\sigma}{\partial \eta}\right).$$

Hence

$$G_n^\sigma I_n^\sigma = (-1)^n \frac{(2n)!}{2^{2n}(n!)^3}\left(\frac{\partial^\sigma}{\partial \xi} + \frac{\partial^\sigma}{\partial \eta}\right)\frac{\partial^{n-\sigma}}{\partial z} V_n$$

$$= (-1)^{n+\sigma}\frac{(2n)!\,(2\sigma)!}{2^{2n+2\sigma}(n!)^3\,\sigma!}(\xi^\sigma + \eta^\sigma)\frac{1}{\gamma}\frac{\partial^{n-\sigma}}{\partial z}\int_{-\gamma}^{\gamma}\frac{\left(1 - \frac{u^2}{\gamma^2}\right)^n du}{\{x^2 + y^2 + (z-u)^2\}^{\sigma + 1/2}}.$$

Enter on the left-hand side of this result, viz., in $G_n^\sigma I_n^\sigma$, the first form of G_n^σ given in § 35; cut out $\xi^\sigma + \eta^\sigma$; and then put $x, y = 0$.

We then obtain

$$\frac{2^{n-\sigma}(n+\sigma)!\,n!}{(2n)!\,\sigma!}\gamma^{n-\sigma}\chi_n^\sigma\left(\frac{z}{\gamma}\right) I_n^\sigma = (-1)^{n+\sigma}\frac{(2n)!\,(2\sigma)!}{2^{2n+2\sigma}(n!)^3\,\sigma!}\frac{1}{\gamma^{2n+1}}\frac{d^{n-\sigma}}{dz}\int_{-\gamma}^{\gamma}\frac{(\gamma^2 - u^2)^n}{(z-u)^{2\sigma+1}}\,du.$$

If we write herein $u = \gamma v$ and put $z = \mu\gamma$, then the limits of the integral I_n^σ will be ∞ and ϵ, where $\gamma^2 + \epsilon = z^2$, and we shall have

$$\chi_n^\sigma(\mu)\, I_n^\sigma = \frac{\{(2n)!\}^2}{2^{3n+\sigma}(n!)^4}\frac{1}{\gamma^{2n+1}}\int_{-1}^{1}\frac{(1-v^2)^n}{(\mu - v)^{n+\sigma+1}}\,dv.$$

When $\sigma = 0$

$$P_n(\mu)\, I_n = \frac{\{(2n)!\}^2}{2^{3n}(n!)^4}\frac{1}{\gamma^{2n+1}}\int_{-1}^{1}\frac{(1-v^2)^n}{(\mu - v)^{n+1}}\,dv.$$

39. The symbol I_n^σ was originally introduced as an abbreviation for

$$\int_\epsilon^\infty \frac{d\theta}{(\theta - \theta_1)^2 \ldots \sqrt{\{(a^2+\theta)(b^2+\theta)(c^2+\theta)\}}}.$$

In the case of prolate spheroids, in which $a = b$, the unequal values of θ are given by

$$\frac{d^\sigma}{d\mu} \mathrm{P}_n(\mu) = \frac{(2n)!}{2^n n! \, (n-\sigma)!} \left\{ \mu^{n-\sigma} - \frac{(n-\sigma)(n-\sigma-1)}{2(2n-1)} \mu^{n-\sigma-2} + \ldots \right\} = 0,$$

in which $(c^2 - a^2)\mu^2 = c^2 + \theta$.

If $n - \sigma$ be even, there will be $\frac{1}{2}(n-\sigma)$ unequal values of θ and either $\frac{1}{2}\sigma$ values $= -a^2$, or $\frac{1}{2}(\sigma - 1)$ along with a factor x or y in the harmonic.

If $n - \sigma$ be odd, there will be $\frac{1}{2}(n - \sigma - 1)$ unequal values of θ with a factor z and either $\frac{1}{2}\sigma$ values $= -a^2$, or $\frac{1}{2}(\sigma - 1)$ with a factor x or y. All cases, however, are included in

$$\mathrm{I}_n^\sigma = \int_\epsilon^\infty \frac{d\theta}{(\theta + a^2)^\sigma (\theta - \theta_p)^2 \ldots (\theta + a^2)\sqrt{(\theta + c^2)}},$$

where $\theta_p, \ldots$ are the unequal values of θ, and, therefore,

$$(\theta - \theta_p)^2 \ldots = \left\{ \frac{2^n n! \, (n-\sigma)!}{(2n)!} \frac{d^\sigma}{d\lambda} \mathrm{P}_n(\lambda) \right\}^2 \gamma^{2n-2\sigma}.$$

In this expression $c^2 + \theta = \lambda^2\gamma^2$, and therefore $a^2 + \theta = (\lambda^2 - 1)\gamma^2$. Hence, recollecting that $c^2 + \epsilon = z^2 = \mu^2\gamma^2$, we obtain for I_n^σ the value

$$\left\{ \frac{(2n)!}{2^n n! \, (n-\sigma)!} \right\}^2 \frac{2}{\gamma^{n+1}} \int_\mu^\infty \frac{d\lambda}{(\lambda^2 - 1)^{\sigma+1} \left\{ \dfrac{d^\sigma}{d\lambda} \mathrm{P}_n(\lambda) \right\}^2}$$

or

$$\left\{ \frac{(2n)!}{2^{n+\sigma} n! \, n!} \right\}^2 \frac{2}{\gamma^{n+1}} \int_\mu^\infty \frac{d\lambda}{(\lambda^2 - 1)^{\sigma+1} \{\chi_n^\sigma(\lambda)\}^2}.$$

Substituting this value in the expressions found in § 38, we find

$$\chi_n^\sigma(\mu) \int_\mu^\infty \frac{d\lambda}{(\lambda^2 - 1)^{\sigma+1} \{\chi_n^\sigma(\lambda)\}^2} = \frac{1}{2^{n-\sigma+1}} \int_{-1}^1 \frac{(1 - v^2)^n}{(\mu - v)^{n+\sigma+1}} dv$$

and

$$\mathrm{P}_n(\mu) \int_\mu^\infty \frac{d\lambda}{(\lambda^2 - 1)\{\mathrm{P}_n(\lambda)\}^2} = \frac{1}{2^{n+1}} \int_{-1}^1 \frac{(1 - v^2)^n}{(\mu - v)^{n+1}} dv.$$

40. These results, for spherical harmonics of the second kind, are probably in the simplest forms for determining the expansions in powers of $1/\mu$. To determine, however, the expression in finite terms, take for simplicity the case of the zonal harmonic, and write $\mu - v = \xi$; then

$$\begin{aligned}(v^2 - 1)^n &= \{(\mu - \xi)^2 - 1\}^n \\ &= (\mu^2 - 1)^n + \ldots + \frac{(-1)^r}{r!} \frac{d^r}{d\mu} (\mu^2 - 1)^n \, . \, \xi^r + \ldots\end{aligned}$$

Dividing by ξ^{n+1} we see that the part of the integral which produces the logarithm is

$$\frac{1}{2^{n+1}n!}\frac{d^n}{d\mu}(\mu^2-1)^n \cdot \int_{\mu-1}^{\mu+1}\frac{d\xi}{\xi},$$

or,

$$\tfrac{1}{2}\,\mathrm{P}_n(\mu)\log\frac{\mu+1}{\mu-1}.$$

The other terms may be grouped in pairs, viz.:—

$$(-1)^{n-s}\frac{d^{n-s}}{d\mu}(\mu^2-1)^n\int\xi^{-s}\,d\xi+(-1)^{n+s}\frac{d^{n+s}}{d\mu}(\mu^2-1)^n\int\xi^s\,d\xi.$$

Hence, by RODRIGUES' theorem (FERRERS, II., § 8) the resulting integration of the pair is

$$(-1)^{n+s}\frac{d^{n+s}}{d\mu}(\mu^2-1)^n\left\{\frac{(\mu+1)^{s+1}-(\mu-1)^{s+1}}{s+1}+\frac{(\mu^2-1)\left[(\mu+1)^{s-1}-(\mu-1)^{s-1}\right]}{s-1}\right\}.$$

41. The last grouping, and the forms of the expressions before integration suggest, in the general case, the substitution

$$\mu-v=\sqrt{(\mu^2-1)}e^{-\eta},$$

so that

$$dv/(\mu-v)=d\eta,$$

and

$$\frac{1-v^2}{\mu-v}=2\{\mu-\sqrt{(\mu^2-1)}\cosh\eta\},$$

while, for

$$v=\pm 1,\quad e^{\pm\eta}=\sqrt{\left(\frac{\mu+1}{\mu-1}\right)},\quad \text{say } e^{\pm\eta_1}.$$

Hence,

$$\frac{1}{2^{n-\sigma+1}}\int_{-1}^{1}\frac{(1-v^2)^n}{(\mu-v)^{n+\sigma+1}}\,dv,$$

which we may denote by T_n^σ

$$=\frac{2^{\sigma-1}}{(\mu^2-1)^{\sigma/2}}\int_{-\eta_1}^{\eta_1}\{\mu-\sqrt{(\mu^2-1)}\cosh\eta\}^n e^{\sigma\eta}\,d\eta$$

$$=\frac{2^\sigma}{(\mu^2-1)^{\sigma/2}}\int_0^{\eta_1}\{\mu-\sqrt{(\mu^2-1)}\cosh\eta\}^n\cosh\sigma\eta\,d\eta.$$

This integral is given in HEINE's 'Kugelfunctionen' (vol. 1, p. 224), and can be readily transformed into others by HEINE's methods, founded on JACOBI's transformations for harmonics of the first kind. Thus, if $\cosh\eta=z$, then

$$\cosh\sigma\eta=\frac{1}{1.3\ldots(2\sigma-1)}\frac{d^\sigma}{dz}(z^2-1)^{\sigma-\frac{1}{2}}\sin h\eta.$$

Transforming then the above integral from η to z, we obtain

$$\frac{2^\sigma}{(\mu^2-1)^{\sigma/2}}\frac{1}{1.3\ldots(2\sigma-1)}\int_1^{z_1}\{\mu - z\sqrt{(\mu^2-1)}\}^n \frac{d^\sigma}{dz}(z^2-1)^{\sigma-\frac{1}{2}}\,dz,$$

where $z_1 = \mu/\sqrt{(\mu^2-1)}$.

Performing σ integrations by parts we obtain

$$\frac{n!}{(n-\sigma)!}\frac{2^\sigma}{1.3\ldots(2\sigma-1)}\int_1^{z_1}\{\mu - z\sqrt{(\mu^2-1)}\}^{n-\sigma}(z^2-1)^{\sigma-\frac{1}{2}}dz.$$

Now write

$$\mu - z\sqrt{(\mu^2-1)} = \frac{1}{\mu+\sqrt{(\mu^2-1)}\cosh\zeta},$$

observing that

$$dz = \frac{\sin h\zeta\, d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^2} \quad \text{and} \quad z^2-1 = \frac{\sin^2 h\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^2}.$$

The integral T_n^σ then becomes

$$\frac{n!}{(n-\sigma)!}\frac{2^\sigma}{1.3\ldots 2\sigma-1}\int_0^\infty \frac{\sin^{2\sigma}h\zeta\,d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^{n+\sigma+1}}$$

$$= \frac{n!}{(n-\sigma)!}\frac{2^\sigma}{1.3\ldots(2\sigma-1)}\int_1^\infty \frac{(z^2-1)^{\sigma-\frac{1}{2}}dz}{\{\mu+\sqrt{(\mu^2-1)}z\}^{n+\sigma+1}}$$

$$= \frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!}\frac{2^\sigma}{1.3\ldots(2\sigma-1)}(\mu^2-1)^{-\sigma/2}\int_1^\infty \frac{\frac{d^\sigma}{dz}(z^2-1)^{\sigma-\frac{1}{2}}.\,dz}{\{\mu+\sqrt{(\mu^2-1)}z\}^{n+1}}$$

$$= \frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!}2^\sigma(\mu^2-1)^{-\sigma/2}\int_0^\infty \frac{\cosh\sigma\zeta\,.\,d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^{n+1}}.$$

(See HEINE's 'Kugelfunctionen,' vol. 1, p. 223).

The zonal harmonic of the second kind takes the comparatively simple form

$$\int_0^\infty \frac{d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^{n+1}}.$$

42. Combining the expression for $\chi_n^\sigma(\mu)\,I_n^\sigma$ given in § 38 with the result of the preceding article, we have

$$(\mu^2-1)^{\sigma/2}\chi_n^\sigma(\mu)\,I_n^\sigma = \frac{(2n)!\,(2n)!\,\gamma^{-2n-1}}{2^{2n+\sigma-1}\,n!\,n!\,(n+\sigma)!\,(n-\sigma)!}\int_0^\infty \frac{\cosh\sigma\zeta\,d\zeta}{\{\mu+\sqrt{(\mu^2-1)}\cosh\zeta\}^{n+1}}$$

where $\mu\gamma = \sqrt{(\gamma^2+\epsilon)}$.

Now $\mathfrak{G}_n^\sigma = G_n^\sigma I_n^\sigma$, and this by §§ 35 and 36 will be found equal to

$$\gamma^n \,.\, 2\cos\sigma\phi \,.\, \frac{2^{n+\sigma}\, n!\, n!}{(2n)!}(1+\mu_1^2)^{\sigma/2}\chi_n^\sigma(\mu_1)\,(\mu^2-1)^{\sigma/2}\chi_n^\sigma(\mu)\, I^\sigma{}_n,$$

$$\text{(by § 41)} \quad = \frac{2^{-n+1}(2n)!}{(n+\sigma)!\,(n-\sigma)!}\, 2\cos\sigma\phi\,(1-\mu_1^2)^{\sigma/2}\chi_n^\sigma(\mu_1)\int_0^\infty \frac{\cosh\sigma\zeta\, d\zeta}{\{\sqrt{(\gamma^2+\epsilon)}+\sqrt{\epsilon}\cosh\zeta\}^{n+1}}$$

$$= \beta_n^\sigma \,.\, 2\cos\sigma\phi \,.\, f_n^\sigma(v')\, F_n^\sigma(\epsilon), \text{ suppose,}$$

where

$$\beta_n^\sigma = (-j)^\sigma \frac{2^{-n-\sigma+1}(2n)!}{n!\,n!};$$

$f_n^\sigma(v')$ is defined in § 36; and

$$F_n^\sigma(\epsilon) = \int_0^\infty \frac{\cosh\sigma\zeta\, d\zeta}{\{\sqrt{(\gamma^2+\epsilon)}+\sqrt{\epsilon}\cosh\zeta\}^{n+1}}.$$

Harmonics of Oblate Spheroids.

43. *Internal Harmonics.*—If we put $a^2 - c^2 = \alpha^2$, and in place of the harmonic functions $\chi_n^\sigma(\lambda)$, $P_n(\lambda)$ we substitute two others ${}_1\chi_n^\sigma(\lambda)$ and ${}_1P_n(\lambda)$ the same as the former, except that the sign of λ^2 is everywhere changed unless in the single factor λ should the degree be odd, then the internal harmonics of oblate spheroids are precisely of the same forms as those given in § 35, i. and ii., provided we substitute α for γ, ${}_1\chi_n^\sigma$ for χ_n^σ, and ${}_1P_n$ for P_n.

In § 36 the internal harmonic for a prolate spheroid was expressed in terms of spheroidal coordinates. That article would require to be recast for the oblate spheroid. Without, however, entering upon the details, I shall merely state the result, viz.:—

$$G_n^\sigma = \alpha_n^\sigma \,.\, 2\cos\sigma\phi\, f(\epsilon) f_1(v),$$

where

$$\alpha_n^\sigma = j^\sigma 2^{n-\sigma}\frac{\{(n+\sigma)!\,(n-\sigma)!\}^2}{(2n)!\,n!\,n!},$$

$$f(\epsilon) = \frac{1}{\pi}\int_0^\pi \{\sqrt{(c^2+\epsilon)} - \sqrt{(a^2+\epsilon)}\cos\theta\}^n \cos\sigma\theta\, d\theta,$$

$$f_1(v) = \frac{1}{\pi}\int_0^\pi \{\sqrt{(-c^2-v)} - \sqrt{(-a^2-v)}\cos\theta\}^n \cos\sigma\theta\, d\theta.$$

44. *External Harmonics.*—If $\alpha^2\mu^2 = \epsilon$ we shall have

$${}_1\chi_n^\sigma(\mu)\, I_n^\sigma = \frac{(2n!)\,(2n)!}{2^{3n+\sigma}\,(n!)^4}\,\frac{1}{\alpha^{2n+1}}\int_{-1}^{1}\frac{(1-v^2)^n}{(\mu-jv)^{n+\sigma+1}}\,dv.$$

I have deduced this expression from applying the operator

$$\frac{(-1)^n}{2^n n!}\frac{(2n)!}{2^n n!\, n!}\left(\frac{\partial^\sigma}{\partial \xi}+\frac{\partial^\sigma}{\partial \eta}\right)\frac{\partial^{n-\sigma}}{\partial z}$$

to the expression for V_n given in § 17, viz., in this case :—

$$\int_\epsilon^\infty \left(1-\frac{z^2}{\lambda}-\frac{\xi\eta}{\alpha^2+\lambda}\right)^n \frac{d\lambda}{(\alpha^2+\lambda)\sqrt{\lambda}},$$

and equating the result to the expression for $G_n^\sigma I_n^\sigma$, then cutting out $\xi^\sigma+\eta^\sigma$ and making $x, y=0$.

It may be observed that the formula of this article might also with great ease be deduced from the expression for V_n noticed in § 21, viz. :—

$$V_n=\int_0^1 (1-\theta^2)^n \left(e^{\theta D}+e^{-\theta D}\right)\frac{1}{\sqrt{(x^2+y^2+z^2)}}\, d\theta ,$$

or,

$$\int_{-1}^1 (1-\theta^2)^n e^{\theta D}\frac{1}{\sqrt{(x^2+y^2+z^2)}}\, d\theta$$

$$=\int_{-1}^1 \frac{(1-\theta^2)^n}{\sqrt{\{x^2+y^2+(z+j\alpha\theta)^2\}}}\, d\theta$$

As, however, this was only established in cases where $x^2+y^2+z^2>\alpha^2$, the above result would not necessarily have been true for points where this condition was not fulfilled.

45. It will be sufficient to state the result of applying the method of this paper to the following expression for V_n, viz. :—

$$V_n=\frac{1}{\pi\alpha^2}\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\iint\left(1-\frac{R^2}{\alpha^2}\right)^{n-\frac{1}{2}}\frac{df\,dg}{\sqrt{\{(f-x)^2+(g-y)^2+z^2\}}}$$

R^2 being put for f^2+g^2.

We thus obtain

$$\chi_n^\sigma\left(\frac{z}{\alpha}\right) I_n^\sigma=(-1)^{n+\sigma}\frac{1}{2^{n+\sigma-1}}\frac{(2n)!}{n!\,n!\,(n+\sigma)!}\frac{1}{\alpha^{n-\sigma+2}}\frac{d^{n+\sigma}}{dz}\int_0^\alpha \frac{\left(1-\frac{R^2}{\alpha^2}\right)^{n-\frac{1}{2}} R\,dR}{\sqrt{(z^2+R^2)}}$$

or, by putting $R=\alpha\sin\theta$, $z=\alpha\mu$,

$$\chi_n^\sigma(\mu)\, I_n^\sigma=\frac{1}{2^{n+\sigma-1}}\frac{(2n)!}{n!\,n!}\frac{1}{\alpha^{2n+1}}\frac{1}{\pi}\int_0^\pi\int_0^\pi \frac{\cos^{2n}\theta\sin\theta\, d\theta\, d\psi}{(\mu+j\sin\theta\cos\psi)^{n+\sigma+1}}.$$

The formulæ of this and the preceding article include, as particular cases, expressions for ${}_1P_n(\mu)\, I_n^\sigma$.

46. The value found for ${}_1\chi_n^\sigma(\mu)\, \mathrm{I}_n^\sigma$ in § 44 leads by the same process as in § 39 to

$$ {}_1\mathrm{T}_n^\sigma = {}_1\chi_n^\sigma(\mu) \int_\mu^\infty \frac{d\lambda}{(\lambda^2+1)^{\sigma+1}\{{}_1\chi_n^\sigma(\lambda)\}^2} = \frac{1}{2^{n-\sigma+1}} \int_{-1}^1 \frac{(1-v^2)^n}{(\mu - jv)^{n+\sigma+1}}\, dv. $$

$$ {}_1\mathrm{P}_n(\mu) \int_\mu^\infty \frac{d\lambda}{(\lambda^2+1)\{{}_1\mathrm{P}_n(\lambda)\}^2} = \frac{1}{2^{n+1}} \int_{-1}^1 \frac{(1-v^2)^n}{(\mu - jv)^{n+1}}\, dv. $$

47. The function ${}_1\mathrm{T}_n^\sigma$ may be expressed as an integral analogous to that in § 41, viz.:—

$$ \frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!}\, 2^\sigma (\mu^2+1)^{-\sigma/2} \int_0^\infty \frac{\cosh\sigma\zeta\, d\zeta}{\{\mu + \sqrt{(\mu^2+1)}\cosh\zeta\}^{n+1}} $$

(HEINE's 'Kugelfunctionen,' vol. 2, p. 129).

48. The external harmonic may also be expressed in terms of spheroidal coordinates in the form

$$ \mathfrak{S}_n^\sigma = \beta_n^\sigma \,.\, 2\cos\sigma\phi \,.\, \mathrm{F}(\epsilon)\, f_1(v), $$

where

$$ \beta_n^\sigma = j^\sigma \frac{2^{-n-\sigma+1}(2n)!}{n!\,n!}, $$

$f_1(v)$ is defined in § 43, and

$$ \mathrm{F}(\epsilon) = \int_0^\infty \frac{\cosh\sigma\zeta}{\{\sqrt{\epsilon} + \sqrt{(\alpha^2+\epsilon)}\cosh\zeta\}^{n+1}}. $$

Expressions for the Reciprocal of the Distance between Two Points.

49. The meanings of H_n^σ and G_n^σ having been changed, the expansion of the reciprocal of the distance between an outside point fgh and an inside xyz requires re-statement.

Referring to § 34, series i. and ii., and to the remark made in § 38, we have

$$ \mathrm{H}_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \mathrm{H}_n^\sigma(x, y, z) = \frac{(2n)!}{2^n n!\,n!} \frac{(n+\sigma)!}{2^{2\sigma} n!\,\sigma!} \left(\frac{\partial^\sigma}{\partial\xi} + \frac{\partial^\sigma}{\partial\eta}\right) \frac{\partial^{n-\sigma}}{\partial z} (\xi^\sigma + \eta)\, z^{n-\sigma} $$

$$ = \frac{(2n)!}{2^n n!\,n!} \frac{(n+\sigma)!\,(n-\sigma)!}{2^{2\sigma-1} n!}; $$

therefore, by § 9

$$ \iint (\mathrm{H}_n^\sigma)^2 ds = \frac{4\pi}{2n+1} \frac{(n+\sigma)!\,(n-\sigma)!}{2^{2\sigma-1}\, n!\,n!}. $$

The required expansion is therefore

$$ \tfrac{1}{2}\mathrm{I}_0(f, g, h) + \ldots + 2^{2\sigma-2}(2n+1) \frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!} \mathrm{G}_n^\sigma(x, y, z)\, \mathrm{G}_n^\sigma(f, g, h) + \ldots $$

in which may be entered any of the foregoing forms investigated for inside and outside harmonics.

By §§ 36 and 42, for example, this expansion in the case of the prolate spheroid for $\epsilon_1 v_1' \phi_1$ and an inner point $\epsilon_2 v_2' \phi_2$ may be thrown into the form:—

$$\mathrm{F}_0(\epsilon_1) + \ldots$$
$$+ (2n+1)\frac{(n+\sigma)!\,(n-\sigma)!}{n!\,n!}\, 2\cos\sigma(\phi_1-\phi_2) f_n^{\sigma}(v_1') f_n^{\sigma}(v_2') f_n^{\sigma}(\epsilon_2)\mathrm{F}_n(\epsilon_1) + \ldots$$

Cylindrical Harmonics.

50. The harmonics pertaining to an elliptic cylinder may be derived from those of the ellipsoid. I shall, however, confine myself to the case of the circular cylinder, which is a limiting form of the prolate spheroid.

Referring to § 36 and observing that when n becomes infinite $(n+\sigma)!\,(n-\sigma)! : n!\,n!$ is ultimately a ratio of equality, and that $2^{2n}\,n!\,n!\,/\,(2n)!$ is ultimately $\surd(\pi n)$, if we make $a=0$, $c=\gamma=\infty$, put $\epsilon=\rho^2$, $\gamma^2+v'=z^2$ and assume $n=\gamma\lambda$, we shall have

$$f_n^{\sigma}(\epsilon) = \gamma^n \frac{1}{\pi}\int_0^{\pi} e^{-\lambda\rho\cos\theta}\cos\sigma\theta\,d\theta = \gamma^n\,\mathrm{J}_{\sigma}(j\lambda\rho)$$

$$\mathrm{F}_n^{\sigma}(\epsilon) = \gamma^{-n-1}\int_0^{\infty} e^{-\lambda\rho\cosh\zeta}\cosh\sigma\zeta\,d\zeta = \gamma^{-n-1}\,\mathrm{K}_{\sigma}(j\lambda\rho).$$

In regard to $f(v')$ let us suppose n and σ both even or both odd, then comparing this integral with the expressions ii. and iv. in § 34, and inverting the series ii. we may put

$$f_n^{\sigma}(v') = \mathrm{A}\left\{1 - \frac{(n-\sigma)(n+\sigma+1)}{1.2}\frac{z^2}{\gamma^2} + \frac{(n-\sigma)(n-\sigma-2)(n+\sigma+1)(n+\sigma+3)}{1.2.3.4}\frac{z^4}{\gamma^4}\,\&\text{c.}\right\}$$

where

$$\mathrm{A} = (-j)^n\frac{1}{\pi}\int_0^{\pi}\cos^n\theta\cos\sigma\theta\,d\theta,$$

the limit of which when n is infinite is $\surd(2/n\pi)$.

The function $f(v')$ in this case therefore reduces to

$$\gamma^n\sqrt{\frac{2}{\pi n}}\cos\lambda z.$$

In like manner, if $n-\sigma$ be odd, we should find

$$\gamma^n\sqrt{\frac{2}{\pi n}}\sin\lambda z.$$

Substituting these various expressions in the formula of § 49 we find for the general term

$$\frac{8}{\pi\gamma}\cos\sigma(\phi_1-\phi_2)\cos\lambda z_1\cos\lambda z_2\,\mathrm{K}_\sigma(j\lambda\rho_1)\,\mathrm{J}_\sigma(j\lambda\rho_2)$$

or, since the origin is arbitrary,

$$\frac{4}{\pi\gamma}\,2\cos\sigma(\phi_1-\phi_2)\cos\lambda(z_1-z_2)\,\mathrm{K}_\sigma(j\lambda\rho_1)\,\mathrm{J}_\sigma(j\lambda\rho_2).$$

Since $n=\gamma\lambda$, if we suppose $n+1=\gamma(\lambda+\partial\gamma)$ we shall have $+1/\gamma=\partial\lambda$, and the reciprocal of the distance between the two points is

$$\Sigma\frac{4}{\pi}\,2\cos\sigma(\phi_1-\phi_2)\int_0^\infty\cos\lambda(z_1-z_2)\,\mathrm{K}\,(j\lambda\rho_1)\,\mathrm{J}\,(j\lambda\rho_2)\,d\lambda.$$

When $\sigma=0$, the term must be divided by 2.

Paraboloidal Harmonics.

51. Any given harmonic of the circular cylinder, considered as a limiting case of a harmonic of the prolate spheroid, is such that, of the characteristics θ in § 3 pertaining to it, some are infinite and others finite. For the infinite values the quantity $\gamma^2+\theta$ may be infinite or finite, but there will be an infinite number of each variety, and the total number of θ's is, as we have seen, of the same order of infinity as γ.

The characteristics which pertain to any given paraboloidal harmonic are in like manner partly finite and partly infinite, there being an infinite number of each, but the order of infinity of the total number is of the same order as $\sqrt{\gamma}$.

By moving the origin of coordinates from the centre to the lower focus of the prolate spheroid, the axis of which is supposed to be vertical, and putting $\epsilon=2\gamma\epsilon'$, $v'=-2\gamma v'$, then making γ infinite, we find for the equation of any paraboloid of the system

$$x^2+y^2=4\epsilon'(\epsilon'+z),$$

and for the orthogonals

$$x^2+y^2=4v'(v'-z).$$

The forms of the harmonics may be readily deduced from the expressions for G_n^σ and $\mathfrak{G}_n^\sigma$ contained in §§ 36 and 42. Thus, when γ is large, we may put

$$f(\epsilon)=f(2\gamma\epsilon')=\gamma^n\frac{1}{\pi}\int_0^\pi\left(1-\sqrt{\frac{2\epsilon'}{\gamma}}\cos\theta\right)^n\cos\sigma\theta\,d\theta,$$

or, if we write herein $\kappa^2\gamma=2n^2$, the expression ultimately becomes

$$\gamma^n\frac{1}{\pi}\int_0^\pi e^{-\kappa\sqrt{\epsilon'}\cos\theta}\cos\sigma\theta\,d\theta=\gamma^n\,\mathrm{J}_\sigma(j\kappa\sqrt{\epsilon'}).$$

Similarly

$$f(v) = \gamma^n \mathrm{J}_\sigma(\kappa\sqrt{v'}),$$

and

$$\mathrm{F}(\epsilon) = \gamma^{-n-1}\int_0^\infty e^{-\kappa\sqrt{\epsilon'}\cosh\zeta}\cosh\sigma\zeta\, d\zeta,$$
$$= \gamma^{-n-1}\mathrm{K}_\sigma(j\kappa\sqrt{\epsilon'}).$$

The expansion for the reciprocal of the distance becomes

$$\Sigma\, 4\cos\sigma(\phi_1 - \phi_2)\frac{n}{\gamma}\mathrm{J}_\sigma(j\kappa\sqrt{\epsilon_2'})\,\mathrm{J}_\sigma(\kappa\sqrt{v_2'})\,\mathrm{K}_\sigma(j\kappa\sqrt{\epsilon_1'})\,\mathrm{J}_\sigma(\kappa\sqrt{v_1'}).$$

When $\sigma = 0$, this term must be divided by 2.

Since $\kappa^2\gamma = 2n^2$, we may put $(\kappa + \delta\kappa)^2\gamma = 2(n+1)^2$, and, therefore, $n/\gamma = \frac{1}{2}\kappa\,\delta\kappa$. Hence the required reciprocal is

$$\Sigma\, 2\cos\sigma(\phi_1 - \phi_2)\int_0^\infty \mathrm{J}_\sigma(j\kappa\sqrt{\epsilon_2'})\,\mathrm{J}_\sigma(\kappa\sqrt{v_2'})\,\mathrm{K}_\sigma(j\kappa\sqrt{\epsilon_1'})\,\mathrm{J}_\sigma(\kappa\sqrt{v_1'})\,\kappa\, d\kappa,$$

the term, in which $\sigma = 0$, being divided by 2.

Electrical Capacities.

52. The electrical capacity of a conductor in the form of the surface obtained by inverting a prolate spheroid, with regard to one of its foci, is readily shown to be

$$\frac{\mathrm{R}^2}{2\gamma}\left\{\log\frac{c+\gamma}{c-\gamma} + 3\left(\log\frac{c+\gamma}{c-\gamma} - 2\frac{\gamma}{c}\right) + 5\left(\log\frac{c+\gamma}{c-\gamma} - \frac{6c\gamma}{3c^2-\gamma^2}\right) + \&\mathrm{c.}\right\}.$$

This series is ultimately convergent, but when γ is nearly equal to c it is inappropriate. When $c = \gamma$ we come to paraboloids, and it may be shown from the expression obtained in the preceding article that the capacity of a cardioide of revolution $r = 2a(1 - \cos\theta)$ is

$$4a^2\int_0^\infty \frac{\mathrm{K}_0(j\kappa\sqrt{a})}{\mathrm{J}_0(j\kappa\sqrt{a})}\kappa\, d\kappa.$$

GREEN has worked out completely the case of a body freely charged, somewhat resembling the cardioide of revolution, but with a conical hollow whose vertex is not a cusp, and it will be found that the capacity is very nearly the same as that of a sphere of equal volume. I see no way, however, of evaluating the above integral so as to make a similar comparison, either as regards volume or surface.

The capacity of an anchor-ring without an aperture, the radius of whose axis is a, may be found from GREEN's Function by means of the series in § 50, viz., it is

$$\frac{8a^2}{\pi}\int_0^\infty \frac{\mathrm{K}_0(j\lambda a)}{\mathrm{J}_0(j\lambda a)}\, d\lambda.$$

Expression of any Ellipsoidal Harmonic in terms of the Conjugate System.

53. Let $u = 1/r$, where $r^2 = (x-f)^2 + (y-g)^2 + (z-h)^2$, and let $v = V_n(x-p, y-q, z-r)$ be a potential arising from an ellipsoid with its centre at p, q, r and its axes parallel to the coordinate axes. We are to consider the invariant operator

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^n$$

upon uv; or, if we take the suffix 1 to be put to differential operators upon u only, and the suffix 2 to those on v only, the operation in question is

$$2^n \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right)^n uv.$$

Now this may be represented by a series of terms of the form

$$A_n^\sigma H_n^\sigma\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial z_1}\right) H_n^\sigma\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2}\right) uv$$

where

$$2^n n!\ A_n^\sigma = H_n^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) H_n^\sigma(x, y, z).$$

If we suppose x, y, z put equal to zero after the operations are performed, this result is clearly the same as

$$A_n^\sigma H_n^\sigma\left(\frac{\partial}{\partial f}, \frac{\partial}{\partial g}, \frac{\partial}{\partial h}\right) H_n^\sigma\left(\frac{\partial}{\partial p}, \frac{\partial}{\partial q}, \frac{\partial}{\partial r}\right) \frac{V_n(p, q, r)}{\sqrt{f^2+g^2+h^2}}$$
$$= A_n^\sigma (2n)!\, \frac{H_n^\sigma(f, g, h)}{d^{2n+1}}\, G_n^\sigma(p, q, r), \quad \S\S\ 7 \text{ and } 20,$$

where $d^2 = f^2 + g^2 + h^2$.

Now let the axes of coordinates be transformed in such a way that the new axis of z is in the direction of the radius vector to fgh; the above operator, which is an invariant, can then be thrown into the form

$$\frac{(2n)!}{n!\ n!}\frac{\partial^n}{\partial z_1}\frac{\partial^n}{\partial z_2} + \ldots + \frac{2^{2\sigma}(2n)!}{(n+\sigma)!\ (n-\sigma)!}\left(\frac{\partial^\sigma}{\partial \xi_1}\frac{\partial^\sigma}{\partial \eta_2} + \frac{\partial^\sigma}{\partial \xi_2}\frac{\partial^\sigma}{\partial \eta_1}\right)\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right)^{n-\sigma} + \ldots$$

The new coordinates of fgh are $0, 0, d$; let those of p, q, r be p', q', r', and let the transformed values of u, v be u', v'. We may omit all but the first term of the operator, inasmuch as the others give vanishing values, and the result is

$$\frac{(2n)!}{n!\ n!}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right)^n u'v'$$
$$= \frac{(-1)^n}{d^{n+1}} \cdot \frac{(2n)!}{n!}\left(\frac{\partial}{\partial z_2}\right)^n v',$$

or, returning to the old axes,

$$(-1)^n \cdot \frac{(2n)!}{n!}\,\frac{1}{d^{n+1}}\left(\frac{\partial}{\partial h}\right)^n \mathrm{V}_n(x-p,\ y-q,\ z-r),$$

where

$$\frac{\partial}{\partial h} = \frac{f}{d}\frac{\partial}{\partial x} + \frac{g}{d}\frac{\partial}{\partial y} + \frac{h}{d}\frac{\partial}{\partial z}.$$

We have, finally, putting $x, y, z = 0$,

$$\left(f\frac{\partial}{\partial p} + g\frac{\partial}{\partial q} + h\frac{\partial}{\partial r}\right)^n \mathrm{V}_n(p, q, r) = 2^n\, n!\ n!\ \Sigma \frac{\mathrm{H}_n^{\sigma}(f, g, h)\ \mathfrak{G}_n^{\sigma}(p, q, r)}{\mathrm{H}_n^{\sigma}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\mathrm{H}_n^{\sigma}(x, y, z)}$$

By equating the coefficients of the various powers and products of f, g, h we may obtain any assigned differential coefficient of V_n.

Addition Theorems for Spherical Harmonics of the Second Kind and for BESSEL'S *Functions.*

54. The method of the preceding article may be used for establishing addition theorems for harmonics of the second kind, but before applying it for this purpose it will be convenient to make a few preliminary observations for the purpose of bringing those harmonics under a similar system of definitions as those for harmonics of the first kind. I begin with a geometrical interpretation of

$$\int_0^{\infty}\frac{du}{\mu + \sqrt{\mu^2 - 1}\cosh u},$$

where μ lies between 1 and -1.

Let P be any point whose projection on the axis of z is N, so that $\mathrm{ON} = z$, $\mathrm{PN} = \rho$, $\mathrm{OP} = r$, $\cos \mathrm{NOP} = \mu$; then the potential at P due to a uniform distribution of matter on ON is

$$\int_0^z \frac{d\zeta}{\sqrt{(z-\zeta)^2 + \rho^2}} = \tfrac{1}{2}\log\frac{r+z}{r-z} = \tfrac{1}{2}\log\frac{1+\mu}{1-\mu}.$$

In like manner, if we draw PM at right angles to OP to meet the axis of z in M, we shall find

$$\frac{Q_0}{r} = \int_{OM}^{\infty} \frac{d\zeta}{\zeta\sqrt{(\zeta - z)^2 + \rho^2}} = \frac{1}{r} \cdot \tfrac{1}{2} \log \frac{1 + \mu}{1 - \mu}$$

$$= \frac{1}{r} \int_0^{\mu} \frac{dv}{1 - v^2}$$

$$= \frac{1}{r} \int_0^{\infty} \frac{\mu \, du}{\mu^2 + (1 - \mu^2) \cosh^2 u}, \text{ where } v = \mu \tanh u$$

$$= \frac{1}{2r} \int_0^{\infty} \frac{du}{\mu + j\nu \cosh u} + \frac{1}{2r} \int_0^{\infty} \frac{du}{\mu - j\nu \cosh u}.$$

Also

$$-j\,\tfrac{1}{2}\pi \frac{P_0}{r} = \frac{1}{2r} \int_0^{\infty} \frac{du}{\mu + j\nu \cosh u} - \frac{1}{2r} \int_0^{\infty} \frac{du}{\mu - j\nu \cosh u},$$

therefore

$$\frac{Q_0 - j\,\tfrac{1}{2}\pi P_0}{r} = \frac{1}{r} \int_0^{\infty} \frac{du}{\mu + j\nu \cosh u}.$$

We shall write R_n for $Q_n - j\,\tfrac{1}{2}\pi P_n$, so that

$$\frac{R_0}{r} = \int_0^{\infty} \frac{du}{z + j\rho \cosh u}.$$

Following now the corresponding definitions for harmonics of the first kind, we may write

$$n!\, Q_n = (-1)^n r^{n+1} \left(\frac{\partial}{\partial z}\right)^n \frac{Q_0}{r}$$

$$n!\, Q_n^{\sigma} = (-1)^n r^{n+1} \left(\frac{\partial}{\partial z}\right)^{n-\sigma} \left(\frac{\partial^{\sigma}}{\partial \xi} + \frac{\partial^{\sigma}}{\partial \eta}\right) \frac{Q_0}{r}$$

$$n!\, Q'^{\sigma}_n = (-1)^n r^{n+1} j \left(\frac{\partial}{\partial z}\right)^{n-\sigma} \left(\frac{\partial^{\sigma}}{\partial \xi} - \frac{\partial^{\sigma}}{\partial \eta}\right) \frac{Q_0}{r}.$$

It is obvious that similar definitions also apply to the R-functions.

In carrying out the differential operations indicated we may observe that, where the function to be operated upon does not contain the longitudinal angle ϕ, the following results will be useful, viz.,

$$2^{\sigma} \frac{\partial^{\sigma}}{\partial \xi} = e^{-\sigma\phi j} \left(\frac{\partial}{\partial \rho} - \frac{\sigma - 1}{\rho}\right) \left(\frac{\partial}{\partial \rho} - \frac{\sigma - 2}{\rho}\right) \cdots \left(\frac{\partial}{\partial \rho} - \frac{1}{\rho}\right) \frac{\partial}{\partial \rho}$$

$$= e^{-\sigma\phi j} f_{\sigma}\left(\frac{\partial}{\partial \rho}\right) \text{ suppose,}$$

and

$$2^{\sigma} \frac{\partial^{\sigma}}{\partial \eta} = e^{\sigma\phi j} f_{\sigma}\left(\frac{\partial}{\partial \rho}\right).$$

We then have, readily,

$$P_n^\sigma = j^\sigma 2^{-\sigma} \,.\, 2 \cos \sigma\phi \,.\, p_n^\sigma$$
$$R_n^\sigma = j^\sigma 2^{-\sigma} \,.\, 2 \cos \sigma\phi \,.\, r_n^\sigma,$$

where

$$p_n^\sigma = \frac{1}{\pi}\int_0^\pi \frac{\cos \sigma\theta}{(\mu + j\nu \cos\theta)^{n+1}} d\theta \qquad r_n^\sigma = \int_0^\infty \frac{\cosh \sigma u \, du}{(\mu + j\nu \cosh u)^{n+1}} \,.$$

When $\sigma = 0$, these general results must be divided by 2. There are also corresponding functions with $\sin \sigma\phi$ substituted for $\cos \sigma\phi$.

55. [*March* 4, 1891.—The expressions contained in § 54 may also be readily obtained if we notice that R_0/r is equal to the integral

$$\frac{1}{2}\int_{-\infty}^\infty \frac{du}{z + jx \cosh u + y \sinh u} \,,$$

or,

$$\frac{1}{2}\int_{-\infty}^\infty \frac{du}{z + \frac{1}{2}j\,(\xi e^{-u} + \eta e^u)} \,.$$

Let us find the result, in a series, of the operation denoted by

$$\frac{(-1)^n}{n!}\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}\right)^n \frac{R_0}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A).$$

One method is given in § 53, but inasmuch as the expression (A) is a spherical harmonic in f, g, h we may at once put it equal to

$$\Sigma\, A_\sigma\, H_n^\sigma (f, g, h)\, H_n^\sigma\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \frac{R_0}{r} \,,$$

and determine A_σ as in §§ 10, 49. The result is

$$\Sigma\, 2^{2\sigma - 1} \frac{n!\, n!}{(n+\sigma)!\,(n-\sigma)!} H_n^\sigma (f, g, h) \frac{(-1)^n}{n!}\left(\frac{\partial^\sigma}{\partial \xi^\sigma} + \frac{\partial^\sigma}{\partial \eta^\sigma}\right)\frac{\partial^{n-\sigma}}{\partial z^{n-\sigma}} \frac{R_0}{r} \,.\Big]$$

If now $d_1\mu_1\phi_1$ and $d_2\mu_2\phi_2$ be the polar coordinates of fgh and xyz respectively, the series first found, when the expressions in the preceding article are entered in it, will become

$$\frac{d_1^n}{d_2^{n+1}}\left\{P_n R_n + \ldots + \frac{n!\, n!}{(n+\sigma)!\,(n-\sigma)!} p_n^\sigma r_n^\sigma\, 2\cos\sigma(\phi_1 - \phi_2) + \ldots\right\}.$$

But the operation (A) also gives

$$\frac{d_1^n}{d_2^{n+1}} \frac{1}{2} \int_{-\infty}^{\infty} \frac{(h + j \cosh u \, f + \sinh u \, g)^n}{(z + j \cosh u \, x + \sinh u \, y)^{n+1}} du$$

Hence

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\{\mu_1 + j\nu_1 \cosh (u + j\phi_1)\}^n}{\{\mu_2 + j\nu_2 \cosh (u + j\phi_2)\}^{n+1}} du$$
$$= \mathrm{P}_n(\mu_1)\,\mathrm{R}_n(\mu_1) + \ldots + (-1)^\sigma \frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!} p_n^\sigma(\mu_1)\, r_n^\sigma(\mu_2)\, 2\cos\sigma(\phi_1 - \phi_2) + \ldots$$

We may denote the left-hand side of this identity by $\mathfrak{R}_n$ and put $\mathfrak{R}_n = \mathfrak{Q}_n - j\frac{1}{2}\pi\mathfrak{P}_n$. Then, since

$$\mathfrak{P}_n = \mathrm{P}_n(\mu_1)\,\mathrm{P}_n(\mu_2) + \ldots + (-1)^\sigma \frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!} p_n^\sigma(\mu_1)\, p_n^\sigma(\mu_2)\, 2\cos\sigma(\phi_1 - \phi_2) + \ldots$$

We must also have

$$\mathfrak{Q}_n = \mathrm{P}_n(\mu_1)\,\mathrm{Q}_n(\mu_2) + \ldots + (-1)^\sigma \frac{n!\,n!}{(n+\sigma)!\,(n-\sigma)!} p_n^\sigma(\mu_1)\, q_n^\sigma(\mu_2)\, 2\cos\sigma(\phi_1 - \phi_2) + \ldots$$

56. The symbol $\mathfrak{Q}_n$ has been used to denote the real part of the result of the differential operation

$$\frac{(-1)^n}{n!} r^{n+1} \left(\frac{\partial}{\partial h}\right)^n \int_0^\infty \frac{du}{z + j\rho\cosh u}.$$

To make it clear what this coefficient is and how related to the corresponding function $\mathfrak{P}_n$ it may be noticed that the former, as found in § 53, is at once recognisable as the coefficient of d_1^n/d_2^{n+1} in the expansion of

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{du}{d_2(\mu_2 + j\nu_2\cosh u) - d_1(\mu_1 + j\nu_1\cosh u)},$$

or, of

$$\frac{1}{r_{12}} \left\{ \log \frac{r_{12} + d_2\mu_2 - d_1\mu_1}{r_{12} - d_2\mu_2 + d_1\mu_1} - j\,\tfrac{1}{2}\pi \right\}$$

where

$$r_{12}^2 = (x - f)^2 + (y - g)^2 + (z - h)^2.$$

If we strike out the second term within the brackets we obtain the generating function of $\mathfrak{Q}_n$.

The expansion in the preceding article has been proved in the case when μ_2 is not

greater than 1, that case admitting of a geometrical interpretation, but a similar relation, from its character, must also hold when μ_2 is greater than 1.

Reduction to BESSEL'S *Functions.*

57. If we make n infinite and write $n\nu_1 = \lambda_1$ and $n\nu_2 = \lambda_2$, where λ_1, λ_2 are finite, the left-hand side of the integral in § 55, expressing Q_n, will become

$$\begin{aligned} &\tfrac{1}{2}\int_{-\infty}^{\infty} e^{j\lambda_1 \cosh (u + j\phi_1) - j\lambda_2 \cosh (u + j\phi_2)}\, du, \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} e^{j\Lambda \cosh (u + j\Phi)}\, du, \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} e^{j\Lambda \cosh u}\, du, \end{aligned}$$

where

$$\Lambda \cos \Phi = \lambda_1 \cos \phi_1 - \lambda_2 \cos \phi_2,$$
$$\Lambda \sin \Phi = \lambda_1 \sin \phi_1 - \lambda_2 \sin \phi_2,$$

and, therefore,

$$\Lambda^2 = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos (\phi_1 - \phi_2).$$

Following HEINE, to whose work I am indebted for the integrals expressing R_0 in §§ 54 and 55, I shall denote the external BESSEL'S function by K, that is to say, the limiting form of Q in the same way as J is used as the limiting form of P. The above integral is therefore equal to $K_0(\Lambda) - j\tfrac{1}{2}\pi J_0(\Lambda)$. The series in § 55 then readily give

$$J_0(\Lambda) = J_0(\lambda_1) J_0(\lambda_2) + \ldots + 2J_\sigma(\lambda_1) J_\sigma(\lambda_2) \cos \sigma (\phi_1 - \phi_2) + \ldots$$

$$K_0(\Lambda) = J_0(\lambda_1) K_0(\lambda_2) + \ldots + 2J_\sigma(\lambda_1) K_\sigma(\lambda_2) \cos \sigma (\phi_1 - \phi_2) + \ldots$$